Recurrences

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ABSTRACT. We identify errors in Leighton's *Notes on Better Master Theorems for Divide-and-Conquer Recurrences* and provide counterexamples. Convenient replacements are provided for the main result.

We define *admissibility* of recurrences and prove that a solution T of an admissible recurrence satisfies a strong form of the Akra-Bazzi formula if and only if T is locally $\Theta(1)$, a property implied by bounded depth of recursion on bounded sets, which in turn is implied by satisfaction of a ratio condition on the dependencies of the recurrence. We show as a consequence that if R is a divide-and-conquer recurrence with low noise whose recursion set contains only integers and whose incremental cost satisfies our generalization of Leighton's polynomial-growth condition, then the solution of R satisfies the same strong Akra-Bazzi condition. Generalizations of the Master Theorem and an application to nonhomogeneous linear difference equations are also provided along with some results about asymptotic solution insensitivity to the base case and incremental cost of a recurrence.

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I have to see something to the point where
I have surrounded it and … totally understood it …

— DONALD KNUTH

You always hit obstacles. You get past them because
you just keep thinking about it until it gives up.

— JACOB LURIE

Preface

These notes began life in 2010 as an email never sent to Tom Leighton. The message was a sketchy list of errata for his very interesting paper about the Akra-Bazzi formula. However, I was unable to resist providing a comprehensive explanation and resolution of the issues along with extensive discussion of related topics. Work proceeded as an intermittent back-burner project with many delays.

The intended audience for this lengthy exposition includes anyone interested in solutions of recurrences. A high level of detail is provided. The reader should have Leighton's paper on hand for reference while reading some parts of these notes.

An overview of the main points is provided in the *Introduction.* I recommend at least browsing that section.

Counterexamples to Theorem 2 of the aforementioned paper are exhibited, and errors in the argument are identified. Convenient replacements for that proposition are provided.

Much of the current document is applicable to *admissible* recurrences, which are defined herein. The key result is that a solution of an admissible recurrence satisfies a strong version of the Akra-Bazzi formula if and only if the solution is locally $\Theta(1)$. Both of these properties are consequences of the *bounded depth condition*, which is implied by the *ratio condition.*

The most interesting consequence is Theorem 21.2: If R is a *divide-and-conquer recurrence* with *low noise* whose *recursion set* contains only integers and whose *incremental cost* has *polynomial growth*, then its solution satisfies the *strong Akra-Bazzi condition* relative to R and each *tame* extension of the incremental cost of R. Furthermore, there exist such extensions.

The well-known algorithms book by Cormen, Leiserson, Rivest, and Stein contains a proposition called the *Master Theorem*. I provide generalizations of the Master Theorem in Section 33.

Counterexamples to Leighton's sufficiency criterion for satisfaction of his polynomial-growth condition are provided. An adaptation of that condition to more general domains is extensively analyzed. Roughly speaking, a polynomial-growth function is a non-negative real-valued function on a set of positive real numbers such that the function has bounded dynamic range on subsets of the domain with uniformly bounded dynamic range. I regret my perpetuation of the terminology *polynomial growth* for the phenomenon in question. The relationship to polynomials is far too loose. I should have used some other terminology such as *uniformly constrained dynamic range.*

The main results are stated in Sections 20 and 21. Their proofs are largely contained in Sections 20–28, although there some dependencies on earlier sections. See Section 35 for applications to nonhomogeneous linear difference equations with constant coefficients. Section 29 establishes solution insensitivity to certain changes in the base case and incremental cost of a divide-and-conquer recurrence satisfying mild conditions.

This document suffers from an uneven style partly because it was written sporadically over a long period of time. I hope this causes no confusion for the reader.

There is some redundancy in this work, although not as much as suggested by the great length. In particular, most of the information in the Introduction is repeated in later sections.

I am guilty of a major faux pas: This document was not typeset with TeX. Furthermore, the uppercase letters *I* and *J* are too similar as are the symbols [ ], ⌊ ⌋, ⌈ ⌉, and | |. The reader has my apologies.

Errata and other comments are welcome at the email address on the title page. This document will be updated as appropriate. The latest version can be found at the internet address specified on the copyright page. Let’s get it right!

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Contents

0. Introduction

This work is inspired by Tom Leighton's *Notes on Better Master Theorems for Divide-and-Conquer Recurrences* [Le], which starts by saying

> Divide-and-conquer recurrences are ubiquitous in the analysis of algorithms. Many methods are known for solving recurrences such as
>
> $$T(n) = \begin{cases} 1, & \text{if } n = 1 \\ 2T(\lceil n/2 \rceil) + O(n), & \text{if } n > 1, \end{cases}$$
>
> but perhaps the most widely taught approach is the Master Method that is described in the seminal algorithms text by Cormen, Leiserson and Rivest [the first edition, which was before Stein became a coauthor].
>
> The Master Method is fairly powerful and results in a closed form solution for divide-and-conquer recurrences with a special (but commonly-occurring) form. Recently Akra and Bazzi [AB] discovered a far more general solution to divide-and-conquer recurrences….
>
> In these notes, we give a simple inductive proof of the Akra-Bazzi result … We also show that the Akra-Bazzi result holds for a more general class of recurrences that commonly arise in practice and that are often considered to be difficult to solve.

(The actual citations in [Le] differ superficially from those shown in the quotation above, but they refer to the same sources. Due to limitations of the software used to create this document, the punctuation and formatting of the recurrence above also differ from [Le].)

The description in [CLRS] of the *Master Method* is adapted from work of Jon Bentley, Dorothea Haken, and James Saxe [BHS] and is encapsulated by the *Master Theorem* in [CLRS]. A couple of Leighton's conclusions are included in [CLRS].

Akra-Bazzi Theorem. The main result in [AB] is applicable to recurrences of the form

$$T(n) = \sum_{i=1}^{k} a_i T(\lfloor b_i n \rfloor) + g(n)$$

for each positive integer n where $T(0) > 0$ and there are various assumptions about a_i, b_i, and g. They conclude that

$$T(n) = \Theta\left(n^p\left(1 + \int_{n_1}^{n} \frac{g(u)}{u^{p+1}} du\right)\right)$$

for sufficiently large n_1 where p is determined by

$$\sum_{i=1}^{k} a_i b_i^p = 1.$$

Akra and Bazzi use slightly different notation. For example, their recurrence defines a sequence $u_0, u_1, u_2, \ldots$ rather than a function T, and their b_i is the reciprocal of our b_i. Unlike the depiction above, they organize the right side of their formula as a sum of two terms.

They also state a result analogous to the Master Theorem under the assumptions of their theorem. Their assumptions differ from the hypothesis of the Master Theorem.

Leighton's Theorem 1. Leighton refers to his Theorem 1 as "the Akra-Bazzi result" and gives a simpler proof than the original argument of Akra and Bazzi. Theorem 1 is applicable to recurrences of the form

$$T(x) = \begin{cases} \Theta(1), & \text{for } 1 \le x \le x_0 \\ \displaystyle\sum_{i=1}^{k} a_i T(b_i x) + g(x), & \text{for } x > x_0 \end{cases}$$

that are defined on the real interval $[1, \infty)$ and satisfy a list of assumptions including $a_i > 0$ and $0 < b_i < 1$ for each index i. The function g must be non-negative and satisfy a certain polynomial-growth condition defined by Leighton. The proposition says

$$T(x) = \Theta\left(x^p\left(1 + \int_{1}^{x} \frac{g(u)}{u^{p+1}} du\right)\right)$$

where p is determined by

$$\sum_{i=1}^{k} a_i b_i^p = 1$$

as in [AB].

Leighton's Theorem 1 implies the Akra-Bazzi theorem provided the solution of the recurrence of Theorem 1 is unaffected up to Θ-equivalence when $b_i x$ is replaced by $\lfloor b_i x \rceil$ in the recurrence. This common, highly plausible assumption is true in this instance, as will be evident from our replacements for Theorem 2 of [Le].

In this document, the term *Akra-Bazzi formula* usually refers to the formula in Leighton's Theorem 1, which differs from Akra and Bazzi's version in two respects: the domain is $[1, \infty)$ and the lower limit of integration is always 1.

Leighton's Theorem 2. The most interesting proposition in [Le] is Theorem 2, which is the aforementioned extension of the Akra-Bazzi theorem to "a more general class of recurrences" and is applicable to many recurrences of the form

$$T(x) = \begin{cases} \Theta(1), & \text{for } 1 \le x \le x_0 \\ \displaystyle\sum_{i=1}^{k} a_i T\big(b_i x + h_i(x)\big) + g(x), & \text{for } x > x_0. \end{cases}$$

Theorem 2 assumes there exists $\varepsilon > 0$ such that

$$|h_i(x)| \le \frac{x}{\log^{1+\varepsilon} x}$$

for all $x \ge x_0$ and each index i and has a complicated list of other conditions, including all the assumptions of Theorem 1. The proposition asserts satisfaction of Leighton's version of the Akra-Bazzi formula.

Unfortunately, [Le] contains major and minor errors along with a few oddities. Most significantly, Theorem 2 is false. In this document, we identify the issues and describe suitable resolutions for them. In particular, we obtain convenient replacements for Leighton's Theorem 2.

A finitely recursive counterexample to Theorem 2 is constructed in Section 15. The recurrence's unique solution does not conform to the Akra-Bazzi formula.

Infinitely recursive counterexamples to Theorem 2. Leighton's argument for his Theorem 2 implicitly misidentifies some infinitely recursive recurrences as finitely recursive. In Section 13, we exhibit an infinite family of recurrences partially parameterized by $x_0 \in [686{,}10000]$. They satisfy the hypothesis of Theorem 2 but are infinitely recursive.

Each recurrence in the family has a real-valued solution that maps each non-empty open subset of (x_0, ∞) surjectively onto the set of all real numbers. In particular, each such solution is unbounded above and below on each such open set and has a graph that is dense in the open half plane define by the inequality $x > x_0$. Each recurrence in the family has infinitely many such solutions, but also has a solution that is $\Theta(1)$ as predicted by Leighton's Theorem 2.

Are all solutions real-valued? Like many authors, Leighton does not define solutions of recurrences to be real-valued. He simply describes equations that must be satisfied. All recurrences satisfying the hypothesis of his Theorem 1 are finitely recursive and have unique solutions, which are real-valued. Finitely recursive recurrences satisfying the hypothesis of Theorem 2 also have unique solutions, which are real-valued. However, each of our infinitely recursive counterexamples to Theorem 2 has among its infinitely many solutions some that are not finite. For example, there are solutions $T_{+\infty}$ and $T_{-\infty}$ that satisfy $T_{+\infty}(x) = +\infty$ and $T_{-\infty}(x) = -\infty$ for all $x > x_0$.

With obvious modifications to the arguments in Section 13, it is also possible to show that each of our infinitely recursive counterexamples to Theorem 2 has an infinite number of complex solutions that map each open subset of (x_0, ∞) surjectively onto the complex numbers. In particular, all values of such solutions are finite but some are non-real.

Henceforth, any reference to a "solution" of a recurrence satisfying the hypothesis of Theorem 2 means a "real-valued solution".

Invalid test for Leighton's polynomial-growth condition. [Le] contains the remark "If $|g'(x)|$ is upper bounded by a polynomial in x, then $g(x)$ satisfies the polynomial-growth condition [as defined by Leighton]." The assertion is false, but it is repeated in [CLRS]. We exhibit four classes of differentiable counterexamples in Section 3, where the converse of the remark is also shown to be false for differentiable functions. We also note that in spite of Leighton's remark, differentiability of g is mentioned nowhere in [Le].

Generalization of Leighton's polynomial-growth condition. Among other requirements, functions satisfying Leighton's polynomial-growth condition are non-negative real-valued functions with domains containing certain positive, unbounded intervals. (Non-negativity is not mentioned in Leighton's statement of his condition; however, the condition is stated in the context of a particular class of non-negative functions.) In Section 2, we generalize Leighton's polynomial-growth condition to include all sets of positive real numbers as function domains. Our polynomial-growth functions are also non-negative (indeed, Lemma 2.7 says they are either positive or identically zero). Corollary 2.17 says the two polynomial-growth conditions are equivalent in the context of Leighton's propositions.

Convenient methods for recognizing many functions that satisfy our polynomial-growth condition are provided in Section 4. Subject to very minor restrictions, power functions, non-negative constant functions, logarithms, floors, and ceilings have polynomial growth. Sums, products, quotients (with positive denominators), and compositions of polynomial-growth functions have polynomial growth. Modest perturbations of polynomial-growth functions yield polynomial-growth functions as do certain generalized polynomial functions that satisfy a positivity condition.

Extension of integer recurrences to real intervals. Recurrences arising from the analysis of algorithms are typically defined on a set of integers (usually the positive integers or non-negative integers). Many recurrences defined on sets of integers have natural extensions to recurrences defined on a real interval. For example, the recurrence

$$S(n) = \begin{cases} 1, & \text{for } n = 1 \\ S(\lfloor n/2 \rfloor) + S(\lceil n/2 \rceil) + n, & \text{for each integer } n > 1. \end{cases}$$

defined on the positive integers has the obvious extension

$$T(x) = \begin{cases} 1, & \text{for } x \in [1, 2) \\ T(\lfloor x/2 \rfloor) + T(\lceil x/2 \rceil) + x, & \text{for } x \in [2, \infty) \end{cases}$$

to the real interval $[1, \infty)$. Each of these recurrences is finitely recursive and has a unique solution. Furthermore, S is the restriction of T to the positive integers.

Leighton's Theorem 2 is a statement about a class of recurrences defined on the real interval $[1, \infty)$. Potential applicability to recurrences on the positive integers is via extensions of such recurrences to $[1, \infty)$.

However, the hypothesis of Theorem 2 is applicable to a wider class of recurrences than those obtained by extension from sets of integers, i.e., there is no requirement that $b_i n + h_i(n)$ is an integer for each integer $n > x_0$. Furthermore, the hypothesis very loosely couples behavior of $b_i x + h_i(x)$ outside the integers to behavior on the integers. Therefore, we should not expect such recurrences to behave like integer recurrences. The differences are related to some of the many issues with Theorem 2.

Our definitions of a *divide-and-conquer recurrence* and a *mock* divide-and-conquer recurrence. A *semi-divide-and-conquer recurrence* is of the form

$$T(x) = \begin{cases} f(x), & \text{for } x \in D \backslash I \\ \displaystyle\sum_{i=1}^{k} a_i T\big(b_i x + h_i(x)\big) + g(x), & \text{for } x \in I. \end{cases}$$

The *domain*, D, of the recurrence can be any set of real numbers with a positive element. By definition, a solution T of the recurrence must be a real-valued function on D. The *recursion set*, I, is a non-empty upper subset of D with a positive lower bound. Here k is any positive integer. For each index i, the coefficients a_i and b_i are real numbers with $a_i > 0$ and $0 < b_i < 1$. The *base case*, f, is a real-valued function on $D \backslash I$ with a positive lower bound and a finite upper bound. The *incremental cost*, g, is a non-negative real-valued function on I. Each *noise term* h_i is a real-valued function on I, and $b_i x + h_i(x) \in D$ for all $x \in I$ and each index i. The functions $x \mapsto b_i x + h_i(x)$ are the *dependencies* of the recurrence. The *Akra-Bazzi exponent* of the recurrence is the unique real number p for which

$$\sum_{i=1}^{k} a_i b_i^p = 1.$$

(See Lemma 11.1.) The recurrence is *proper* if $b_i x + h_i(x) < x$ for all $x \in I$ and each index i. A *divide-and-conquer recurrence* is a proper *semi-divide-and-conquer recurrence*. A *mock* divide-and-conquer recurrence is an improper semi-divide-and-conquer recurrence.

A semi-divide-and-conquer recurrence is defined more formally in Section 7 as a $(3k + 4)$-tuple. The formal definition avoids ambiguity in the choice of b_i and h_i.

Relationship of our definitions to Theorem 2. A recurrence satisfying the hypothesis of Theorem 2 also satisfies our definition of a semi-divide-and-conquer recurrence with domain $[1, \infty)$, recursion set $I = (x_0, \infty)$, incremental cost $g|_I$, and noise terms $h_1|_I, \ldots, h_k|_I$ if and only if

$$b_i x + h_i(x) \in [1, \infty)$$

for all $x > x_0$ and each index i. (The domains of g and $h_1, \ldots, h_k$ properly contain I.) It is a divide-and-conquer recurrence if and only if

$$b_i x + h_i(x) \in [1, x)$$

for each such x and i.

As evident from the argument in [Le], the hypothesis of Theorem 2 is intended to imply satisfaction of our definition of a divide-and-conquer recurrence. However, we shall see that this implication is false.

Some mock divide-and-conquer recurrences satisfy the hypothesis of Theorem 2. Members of the aforementioned family of infinitely recursive counterexamples to Theorem 2 are divide-and-conquer recurrences if and only if $x_0 = 10000$.

Each recurrences in the family with $x_0 \neq 10000$, i.e., $x_0 < 10000$ is a mock divide-and-conquer recurrence with infinitely many $x > x_0$ satisfying $b_1 x + h_1(x) > x$, i.e., $T(x)$ depends on $T(y)$ for some $y > x$. Furthermore, 10000 is a fixed point of the sole dependency, i.e., $b_1 \cdot 10000 + h_1(10000) = 10000$.

Our finitely recursive counterexample to Theorem 2 is a divide-and-conquer recurrence. We cannot fix Theorem 2 by simply requiring finite recursion and satisfaction of our definition of a divide-and-conquer recurrence.

Ill posed recurrences. There exist ill posed recurrences (we use the term loosely) that satisfy the hypothesis of Theorem 2 but are neither divide-and-conquer recurrences nor mock divide-and-conquer recurrences. They have the property that $b_i x + h_i(x) < 1$ for some $x > x_0$ and some index i, i.e., $T(x)$ depends on the value of $T(y)$ for some y not in the domain of T. Such a recurrence has no solution, let alone one satisfying the Akra-Bazzi formula. See Section 19 for an example.

Ill-posed recurrences satisfying the hypothesis of Leighton's Theorem 2 can be converted to semi-divide-and-conquer recurrences by extending the domain of the base case to include some values less than 1 while leaving x_0 unchanged. However, issues with the base case of Leighton's inductive argument are exacerbated if we extend the hypothesis of Theorem 2 to include such recurrences. See Section 16.

Since our previously mentioned counterexamples to Theorem 2 are divide-and-conquer recurrences or mock divide-and-conquer recurrences, Theorem 2 cannot be fixed by merely avoiding ill posed recurrences.

Inductive proof of Theorem 2 uses partition of $[1, \infty)$ that does not necessarily have desired relationship to recurrence. Leighton's argument uses a partition of the recurrence's domain, $[1, \infty)$, into subintervals

$$I_0 = [1, x_0], I_1 = (x_0, x_0 + 1], I_2 = (x_0 + 1, x_0 + 2], \ldots$$

and proceeds by induction on the index of the interval. There is an implicit assertion that

$$b_i x + h_i(x) \in \bigcup_{j=0}^{n-1} I_j$$

for each positive integer n, all $x \in I_n$, and all $i \in \{1, \ldots, k\}$. Leighton correctly demonstrates a similar assertion in the context of his Theorem 1. However, the assertion is false in the context of Theorem 2 and is violated by all the aforementioned counterexamples to that proposition. This is a critical error, which exemplifies the differences between integer recurrences and real recurrences.

Proof of Theorem 2 states lower and upper bounds on $T(x)$ for all $x > x_0$. The lower bound is of the form

$$\left(1 + \frac{1}{\log^{\varepsilon/2} x}\right) y(x)$$

and the upper bound is of the form

$$\left(1 - \frac{1}{\log^{\varepsilon/2} x}\right) z(x)$$

Here y and z are certain real-valued functions on $[1, \infty)$ that have positive lower bounds and finite upper bounds on each bounded subset of their domain.

Mismatch between base case of induction and conclusion of inductive argument. Although the proof of Theorem 2 states the goal of establishing the previously mentioned bounds for T on (x_0, ∞), the base case of the induction is identified as $[1, x_0]$, the domain of the base case of the recurrence, which is disjoint from (x_0, ∞). Indeed, the argument for the asserted bounds of T on (x_0, ∞) depends on validity of the bounds on at least part of $[1, x_0]$.

Base case of induction involves division by zero when $x = 1$. The asserted bounds for $T(x)$ involve division by zero when $x = 1$ because $\log^{\varepsilon/2} 1 = 0$. In context, the only plausible interpretations are that $1/(\log^{\varepsilon/2} 1)$ is either undefined or represents $+\infty$, which corresponds to the obviously false chain of inequalities

$$+\infty \le T(1) \le -\infty.$$

An implausible interpretation is that $1/(\log^{\varepsilon/2} 1)$ represents $-\infty$, which corresponds to the trivial chain of inequalities

$$-\infty \le T(1) \le +\infty.$$

Inductive hypothesis also unsatisfied on part of $(1, x_0]$. The hypothesis of Theorem 2 implies the restriction of T to $[1, x_0]$ has a positive lower bound and a finite upper bound. However, the inductive hypothesis's lower bound for T approaches ∞ as x approaches 1 from above, while the inductive hypothesis's upper bound for T approaches $-\infty$. Also, the asserted upper bound is non-positive for all $x \in (1, e]$ while the asserted lower bound is positive for each such x. Behavior near e is also problematic. See Section 16.

Partial resolution of base case of induction. The problems with the base case of Leighton's induction can be avoided for some, but not all, semi-divide-and-conquer recurrences satisfying the hypothesis of Theorem 2 by restricting the base case of the induction to a suitable proper subset of $(e, x_0]$. As explained in Section 16, this mitigation is possible if and only if

$$\inf_{x > x_0} \left(\inf_{1 \le i \le k} \left(b_i x + h_i(x) \right) \right) > e.$$

Specified bounds are sometimes unsatisfied even when conclusion of Theorem 2 is correct. Sections 17 and 18 give examples of finitely recursive divide-and-conquer recurrences that satisfy the hypothesis and conclusion of Theorem 2 but do not satisfy the bounds asserted by the argument in [Le]. A suitably restricted base case of the induction satisfies the inductive hypothesis, but the inductive step fails.

Other invalid inequalities when $p < 0$. The inductive step of the proof of Theorem 2 implicitly asserts a pair of inequalities that are mutually incompatible when $p < 0$ (see Section 24). This is resolved in our analogous Lemma 20.8 by replacement of p by $|p|$ in conditions 4(a) and 4(b) of [Le] (see the *technical condition* in Section 20) and replacement of the incompatible inequalities with alternatives (see Section 25). Mere replacement of p by $|p|$ without other changes is inadequate because the finitely recursive counterexample in Section 15 to Theorem 2 has $p = 0$, i.e., $|p| = p$. The replacement of p by $|p|$ is also useful in the proof of Lemma 20.9.

Lemma 2 of [Le] fails for some divide-and-conquer recurrences. The finitely recursive divide-and-conquer recurrence in Section 15 that is a counterexample to Theorem 2 is also a counterexample to Lemma 2. Our proper, infinitely recursive counterexample to Theorem 2 with $x_0 = 10000$ is also a counterexample to Lemma 2. See Section 19 for details.

Lemma 2 also fails for each mock divide-and-conquer recurrence that satisfies the hypothesis of Theorem 2 and has positive g. Under the same hypothesis as Theorem 2, the lemma implies (among other consequences) that

$$\int_{b_i x + h_i(x)}^{x} \frac{g(u)}{u^{p+1}} du > 0$$

for all $x \geq 1$ and each index i when g is a positive function. However, for each mock divide-and-conquer recurrence that satisfies the hypothesis of Theorem 2, there exists $x > x_0$ such that $x \leq b_i x + h_i(x)$, so the oriented integral is non-positive in violation of Lemma 2. The counterexamples in Section 13 to Theorem 2 have positive g; those recurrences with $x_0 < 10000$ are improper. See Section 19.

Replacement for Lemma 2. An obvious replacement for Lemma 2 is provided in Section 22 and is applicable to divide-and-conquer recurrences that satisfy the *strong ratio condition* and have an incremental cost with a *tame* extension.

Ratio and Strong Ratio Conditions. A semi-divide-and-conquer recurrence satisfies the *ratio condition* if there exists $\beta < 1$ such that $r(x) \leq \beta x$ for each dependency r and each x in the recursion set. (In particular, the recurrence is proper, i.e., is a divide-and-conquer recurrence) A divide-and-conquer recurrence satisfies the *strong ratio condition* if it satisfies the ratio condition and there exists $\alpha > 0$ such that $\alpha x \leq r(x)$ for each dependency r and each x in the recursion set. See Section 9 for more information.

Tame Functions. We define a *tame* function to be a polynomial-growth, locally Riemann integrable, real-valued function on a non-empty positive interval. Tame functions inherit non-negativity from our definition of polynomial growth. (Our tame functions are unrelated to functions called "tame" in the study of Fréchet spaces.)

Missing integrability conditions. Although all four propositions in [Le] have conclusions involving integrals with $g(u)/u^{p+1}$ as the integrand, the paper makes no mention of any integrability conditions or any other conditions (such as continuity of g) that imply integrability of the integrand. We note that all integrals appearing in our counterexamples to Leighton's Theorem 2 have Riemann integrable integrands, so Theorem 2 cannot be fixed by addition of an integrability condition.

Theorem 2 leaves domains of g and $h_1, \ldots, h_k$ unspecified but has conditions describing their behavior on sets properly containing their natural domains. The form of the recurrence implicitly requires the domains to contain (x_0, ∞). Furthermore, solutions of the recurrence are unaffected by behavior of g or $h_1, \ldots, h_k$ outside this interval. However, the Akra-Bazzi formula, Leighton's polynomial-growth condition, and Lemma 2 implicitly require a larger domain of g (as do Lemma 1 and Theorem 1). Condition (3) of Theorem 2 also implies a larger domain of g whenever

$$b_i x + h_i(x) \leq \min(x, x_0)$$

for some $x \geq 1$ and some index i. Conditions (2) and (3) implicitly require a larger domain for $h_1, \ldots, h_k$.

Leighton's second example is wrong and illustrates some oddities. The second example in [Le] incorrectly gives $\Theta(x^2/\log\log x)$ as the solution to a certain family of recurrences. If we assume an appropriate domain and base case, the correct solution is $\Theta(x^2 \log\log x)$ subject to a caveat about asymptotic incremental costs, which we discuss later.

The example also illustrates the awkwardness of Leighton's implicit domain of g and the choice of 1 as the lower limit of integration. His Theorems 1 and 2 are inapplicable to recurrences that obviously have the same solutions as members of the family but are disqualified from membership in the family only by these unnecessary restrictions. An artificial definition of the restriction of g to $[1/2, x_0]$ is required for g to be non-negative, defined at 1, and satisfy Leighton's polynomial-growth condition, while avoiding a divergent improper integral. See Section 6 for more information.

Admissible recurrences. Section 20 provides replacements for Theorem 2 that are applicable to *admissible recurrences*. An admissible recurrence is a semi-divide-and-conquer recurrence with *low noise* whose incremental cost has a tame extension.

Low noise. We define a semi-divide-and-conquer recurrence to have *low noise* if either the recursion set is bounded or for each noise term h there exists $c > 1$ such that

$$|h(x)| = O\left(\frac{x}{\log^c x}\right).$$

This definition of low noise is weaker than Leighton's noise bound. Our constraint is only specified in asymptotic form, whereas his constraint is satisfied on the specific interval $[x_0, \infty)$. Furthermore, his exponent is $1 + \varepsilon$ for some ε that is positive and satisfies four additional conditions. Our main replacements for Theorem 2 place no additional restrictions on c.

Floors, ceilings, and noise. Some divide-and-conquer recurrences have almost linear dependencies of the form $x \mapsto \lfloor bx \rfloor$ or $x \mapsto \lceil bx \rceil$ where $0 < b < 1$. For example, complexity of merge sort is described by a recurrence of the form

$$T(n) = \begin{cases} \Theta(1), & \text{for } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + g(n), & \text{for each integer } n > 1. \end{cases}$$

Observe that

$$bx + (\lfloor bx \rfloor - bx)$$

and

$$bx + (\lceil bx \rceil - bx)$$

are representations of such dependencies in the form dictated by our definition of a divide-and-conquer recurrence. The corresponding noise terms $x \mapsto \lfloor bx \rfloor - bx$ and $x \mapsto \lceil bx \rceil - bx$, respectively, are consistent with low noise.

Akra-Bazzi conditions. Let I be the recursion set of a semi-divide-and-conquer recurrence and suppose g is a tame extension of the recurrence's incremental cost, so the domain of g is a non-empty, positive interval containing I. The *Akra-Bazzi estimate* for the recurrence relative to g is the real-valued function A on I defined by

$$A(x) = x^p \left(1 + \int_{x_0}^{x} \frac{g(u)}{u^{p+1}} du\right),$$

where $x_0 = \inf I$ and p is the Akra-Bazzi exponent. The quantity x_0 is positive by definition of a semi-divide-and-conquer recurrence, so the denominator of the integrand is positive; in particular, the denominator is non-zero. If x_0 is not in the domain of g, the integral above is improper; it is convergent by Corollary 10.3 and Lemma 10.5. The integrand is non-negative because tame functions are non-negative. Therefore, the function A is positive. Furthermore, Lemma 20.2 says A is locally $\Theta(1)$.

A solution T of the recurrence satisfies the *strong Akra-Bazzi condition* (relative to the recurrence and g) if there exist positive real numbers λ_1 and λ_2 such that

$$\lambda_1 A(x) \le T(x) \le \lambda_2 A(x)$$

for all x in I. A solution T satisfies the w*eak Akra-Bazzi condition* (relative to the recurrence and g) if I is unbounded and

$$T(x) = \Theta\big(A(x)\big).$$

The weak Akra-Bazzi condition is similar to the Akra-Bazzi formula that appears in Leighton's propositions, but the integrand is more loosely related to the incremental cost and the lower limit of integration is different. As explained in Section 20, there is considerable flexibility in the choice of the lower limit of integration.

We are more interested in the strong Akra-Bazzi condition than the weak Akra-Bazzi condition. Of course, the strong Akra-Bazzi condition implies the weak Akra-Bazzi condition when the recursion set is unbounded.

Equivalence of strong Akra-Bazzi condition to solution of admissible recurrence being locally $\Theta(1)$. Theorem 20.11, says the following three conditions are equivalent for a solution T of an admissible recurrence:

(1) T is *locally* $\Theta(1)$, i.e., each restriction of T to a bounded set has a positive lower bound and finite upper bound.

(2) T satisfies the strong Akra-Bazzi condition relative to the recurrence and *some* tame extension of the incremental cost.

(3) T satisfies the strong Akra-Bazzi condition relative to the recurrence and *each* tame extension of the incremental cost.

By definition, the incremental cost of an admissible recurrence has at least one tame extension, so condition (3) of Theorem 20.11 implies condition (2), which implies condition (1) since the Akra-Bazzi estimate is locally $\Theta(1)$. The nontrivial part of Theorem 20.11 is the assertion that (1) implies (3).

Some finitely recursive, admissible divide-and-conquer recurrences satisfy the weak Akra-Bazzi condition but have solutions that are not locally $\Theta(1)$. Examples are exhibited in Sections 17 and 18.

Akra-Bazzi conditions are properties of specific solutions, not recurrences. Section 20 describes an example (a member of the family of counterexamples in Section 13 to Theorem 2) of an infinitely recursive, proper, admissible recurrence with a positive constant solution that satisfies the strong and weak Akra-Bazzi conditions. However, the recurrence has infinitely many other solutions that satisfy neither the strong nor weak Akra-Bazzi conditions and are not $\Theta(1)$ on any non-empty open subset of the recursion set, which is an unbounded interval.

Bounded depth condition. Recursion is insufficiently constrained in the hypothesis of Theorem 2 and Lemma 2 even if we assume finite recursion and satisfaction of our definition of a divide-and-conquer recurrence. A semi-divide-and-conquer recurrence satisfies the *bounded depth condition* if the depth of recursion is bounded on bounded sets (see Sections 8 and 9). The existence of any partition of the recurrence's domain with properties similar to those implicitly claimed by Leighton's argument for Theorem 2 requires satisfaction of our weaker *bounded depth condition*. Our counterexamples in Section 13 and 15 to Leighton's Theorem 2 violate the bounded depth condition, so there is no partition with the necessary properties. In particular, the inductive step of the proof fails for those counterexamples. See Section 19.

Each admissible recurrence that satisfies the bounded depth condition has a unique solution, which satisfies the strong Akra-Bazzi condition. Lemma 9.10 implies the recurrence has a unique solution, which is locally $\Theta(1)$. We conclude from Theorem 20.11 that the solution satisfies the strong Akra-Bazzi condition relative to the recurrence and each tame extension of the recurrence's incremental cost (Corollary 20.12).

Finite recursion is insufficient to imply solution of proper admissible recurrence satisfies either Akra-Bazzi condition. The counterexample in Section 15 to Leighton's Theorem 2 is a finitely recursive, proper, admissible recurrence whose unique solution satisfies neither the strong nor weak Akra-Bazzi conditions.

Ratio condition implies strong Akra-Bazzi condition for admissible recurrences. Lemma 9.6 implies every admissible recurrence satisfying the ratio condition also satisfies the bounded depth condition. Therefore, each such recurrence has a unique solution, which satisfies the strong Akra-Bazzi condition relative to the recurrence and each tame extension of the recurrence's incremental cost (Corollary 20.13).

Integer recurrences and the strong Akra-Bazzi condition. Lemma 21.1 says every divide-and-conquer recurrence whose recursion set contain only integers satisfies the bounded depth condition and has a unique solution, which is locally $\Theta(1)$. If such a recurrence R has low noise and its incremental cost has polynomial growth, then Theorem 21.2 says the recurrence is admissible, so its solution satisfies the strong Akra-Bazzi condition relative to R and each tame extension of the incremental cost. See Section 5 for information about polynomial-growth interpolation, which plays a role in the proof of Theorem 21.2.

Relationship of our key result to Leighton's Theorem 2. In Section 20, we define the *modified Leighton hypothesis* for admissible recurrences, which is analogous to conditions (1), (2), and (4) of Leighton's Theorem 2 but with p replaced by $|p|$ in strict versions of conditions (4a) and (4b). We do not require condition 3 of Theorem 2; the polynomial-growth condition satisfied by the incremental cost of an admissible recurrence suffices.

Lemma 26.1 is a proposition about admissible recurrences satisfying the modified Leighton hypothesis and is analogous to the inductive hypothesis in Leighton's argument for Theorem 2. The proof of Lemma 26.1 is very similar to Leighton's argument, but relies on our replacement in Section 22 for Lemma 2 of [Le], a different partition of the recursion set (Lemma 23.2), our replacement in Section 25 for inequalities that fail when $p < 0$ (see Sections 24), and other changes to the inductive hypothesis.

The proof of Lemma 26.1 also uses Lemma 20.9, which says each admissible recurrence satisfying the modified Leighton hypothesis also satisfies the bounded depth and strong ratio conditions and has a unique solution, which is locally $\Theta(1)$.

Lemma 26.1 is used to prove Lemma 20.8, which says the solution of an admissible recurrence satisfying the modified Leighton hypothesis satisfies the strong Akra-Bazzi condition relative to the recurrence and each tame extension of the incremental cost. (Theorem 20.11 is not available at the point we prove Lemma 20.8).

Lemma 20.10 says a locally $\Theta(1)$ solution T of an admissible recurrence R with unbounded recursion set must also be the solution of an auxiliary admissible recurrence S that satisfies the modified Leighton hypothesis. The recurrence S is derived from R by extension of the domain of the base case and restriction of the recursion set. Lemma 20.8 implies T satisfies the strong Akra-Bazzi condition relative to S and each tame extension of the incremental cost of S. Lemma 20.6 says T must then also satisfy the strong Akra-Bazzi condition relative to R and each tame extension of the incremental cost of R. In this fashion, we obtain the proof of our most fundamental proposition, Theorem 20.11, in the case of an unbounded recursion set. The proof is straightforward when the recursion set is bounded.

Adjustment of the base case. Leighton's Theorem 2 says "x_0 is chosen to be a large enough constant" so that condition (4) of the proposition is satisfied but does not prove the existence of such a value. According to a footnote, "Such a constant value of x_0 can be shown to exist using standard Taylor series expansions and asymptotic analysis."

Our analogous assertion is Lemma 20.7, which is proved in Sections 27 and 28. The existence of "sufficiently large x_0" in the context of Leighton's Theorem 2 is a consequence of our Corollaries 27.8 and 27.10.

We note that a change to the value of x_0 is a change to the domain of the base case, which is significant when the bounded depth condition is violated. Our counterexamples to Theorem 2 are sensitive to the choice of x_0. They require that $x_0 \leq 10000$.

Limits of solution sensitivity to base case. Section 29 identifies some conditions under which changes to the base case of a divide-and-conquer recurrences have limited asymptotic effect on the solution (including some changes that violate our definition of a divide-and-conquer recurrence).

Asymptotic incremental cost. Roughly speaking, the examples in [Le] implicitly assume that asymptotic behavior of a divide-and-conquer recurrence with incremental cost g and an unbounded recursion set is unaffected (up to Θ-equivalence) by substitution of g^* for g when g^* is any (presumably non-negative) real-valued function on the recursion set satisfying $g^* = \Theta(g)$. As we explain in section 29, the assumption is not universally true but is valid subject to some mild conditions. Indeed, non-negativity of g^* is not always required.

A caveat to Leighton's asserted solution of a recurrence that does not have low noise. The claimed solution, which does not satisfy the Akra-Bazzi formula, is valid if and only if $x_0 > e^2$. See Section 30.

Relationship of our results to the Master Theorem. In Section 33, we establish generalizations of the Master Theorem as a consequence of our replacements for Leighton's Theorem 2. Section 34 discusses some caveats about the Master Theorem as stated in [CLRS].

Nonhomogeneous difference equations with constant coefficients. In Section 35, we apply our results via a change of variables to some nonhomogeneous recurrences of the form

$$T(n) = \begin{cases} \Theta(1), & \text{for } n \in \{n_0 - 1, \dots, n_0 - k\} \\ \sum_{j=1}^{k} a_j T(n-j) + g(n), & \text{for each integer } n \geq n_0 \end{cases}$$

where n_0 and k are integers with $k > 0$, each a_j is a non-negative real number, and g is a non-negative real-valued function defined at each integer $n \geq n_0$. At least one a_j is non-zero. Such recurrences can be represented as linear difference equations with constant coefficients. The equation is *homogeneous* if g is identically zero; otherwise, the equation is *nonhomogeneous*. Theorem 35.1 says

$$T(n) = \Theta\left(\lambda^n \left(1 + \int_{n_0}^{n} \frac{C(u)}{\lambda^u} du\right)\right)$$

if the function g^* on

$$\{e^n : n \geq n_0 \text{ is an integer}\}$$

that maps e^n to $g(n)$ has polynomial growth. Here λ is the unique positive root of the polynomial

$$x^k - \sum_{j=1}^{k} a_j x^{k-j}$$

and C is a continuous, real-valued extension of g to $[n_0, \infty)$ such that the function $z \mapsto C(\log z)$ on $[e^{n_0}, \infty)$ has polynomial growth. There exists such a C by Lemmas 4.6 and 5.1. Theorem 35.1 also says

$$T(n) = \Theta(\lambda^n)$$

if

$$g(n) = O(\lambda^n / n^{1+\varepsilon})$$

for some $\varepsilon > 0$.

Corollary 35.2 is similar but assumes n_0 is positive, replaces polynomial growth of g^* with polynomial growth of g, and concludes that

$$T(n) = \Theta\left(\lambda^n \left(1 + \int_{n_0}^{n} \frac{G(u)}{\lambda^u} du\right)\right)$$

for each tame extension G of g. The corollary also lists conditions that imply

$$T(n) = \Theta(\lambda^n).$$

Existence of a tame extension is guaranteed by Corollary 5.2.

Some authors call such recurrences *linear recurrence relations with constant coefficients.* We avoid that terminology because of potential confusion over the meaning of *linear*. For example, the paper [AB] of Akra and Bazzi has the title, *On the Solution of Linear Recurrence Equations*, although the subject of the paper is what we call divide-and-conquer recurrences with low noise.

Generating functions are commonly used to solve difference equations. See [Kn], [GKP], [GK], and [Wilf]. In Section 7, we explain the use of linear algebra to solve homogeneous difference equations with constant coefficients. The propositions in Section 35 employ very different methods and are applicable to both homogeneous and nonhomogeneous difference equations.

1. Notation, Terminology, and Other Conventions

$\boldsymbol{Z}$, $\boldsymbol{Z}^+$, $\boldsymbol{R}$, $\boldsymbol{R}^+$, $\boldsymbol{C}$, and $\boldsymbol{N}$ represent the sets of integers, positive integers, real numbers, positive real numbers, complex numbers, and the natural numbers (including zero), respectively. Notice the use of boldface. The symbols Z, R, C, and N may be used to represent mathematical objects other than $\boldsymbol{Z}$, $\boldsymbol{R}$, $\boldsymbol{C}$, and $\boldsymbol{N}$. If x is a function on a set of integers, then x is a sequence with x_n representing $x(n)$.

A function described as *positive*, *non-negative*, or *identically zero* is implicitly real-valued. A set is *positive* if it is contained in $\boldsymbol{R}^+$. Unless stated otherwise, the term *number* means *real number*, and language such as "let $x > 1$" refers to real values. Real powers of positive numbers represent positive values. In the absence of a leading minus sign, the symbol ∞ represents $+\infty$. Except where noted, we do not assume real-valued functions defined on intervals to be differentiable, continuous or integrable.

Intervals. The term *interval* usually refers to a *real interval*, which we define to be a connected subset of the real numbers. An interval is *degenerate* if it is empty or is a singleton, i.e., a set consisting of a single element. (Some authors equate degenerate intervals with real singletons.)

The length of an interval X is denoted by $length(X)$. Of course, $length(\emptyset) = 0$ and $length(X) = \sup X - \inf X$ for each non-empty, bounded interval X. In particular, bounded intervals have finite length. Degenerate intervals have length zero, and positive length is synonymous with non-degeneracy.

A *subinterval* of an interval I is an interval that is a subset of I. A *proper subinterval* of I is a subinterval that is a proper subset of I. (Some authors define a *proper* interval to be an interval of positive length; with such a definition, the meaning of "proper subinterval" becomes ambiguous.)

A *positive, unbounded interval* is a positive interval with no finite upper bound, i.e., an interval of the form (L, ∞) for some real $L \geq 0$ or $[M, \infty)$ for some real $M > 0$. If S is a non-empty positive set, the *minimum positive, unbounded interval containing* S is the intersection of all positive unbounded intervals containing S, i.e.,

$$(S \cap \{\inf S\}) \cup (\inf S\,, \infty).$$

Empty sums. We adopt the convention that the sum of an empty series is zero, e.g.,

$$\sum_{n=0}^{-1} e^n = 0.$$

Riemann integral. Unlike many published definitions of the Riemann integral, we consider a real-valued function f to be Riemann integrable on each singleton interval $[c, c]$ in its domain. Of course,

$$\int_c^c f(x)\,dx = 0.$$

The validity of Lebesgue's criterion for Riemann integrability extends to real-valued functions on real singletons. See Section 10 for details.

A real-valued function f on a non-empty interval I is *locally Riemann integrable* if f is Riemann integrable on all non-empty compact subintervals of I. Local Riemann integrability is indeed a local property: f is locally Riemann integrable if and only if for all $x \in I$ there exists a bounded open interval W containing x such that the closure K of $I \cap W$ is contained in I, and f is Riemann integrable on K. If I is compact, then f is locally Riemann integrable if and only if f is Riemann integrable. Local Riemann integrability does not imply convergence of improper integrals. Our terminology is nonstandard: most authors say "Riemann integrable" where we say "locally Riemann integrable".

Asymptotically related real sets. A set S of real numbers is asymptotically contained in a set T of real numbers if $S \cap V \subseteq T$ for some positive, unbounded interval V. Real sets X and Y are asymptotically equal if $X \cap W = Y \cap W$ for some positive, unbounded interval W. Our definitions imply X and Y are asymptotically equal if and only if each of X and Y is asymptotically contained in the other: Asymptotic equality obviously implies the asymptotic containments. The converse: $X \cap I \subseteq Y$ and $Y \cap J \subseteq X$ for positive, unbounded intervals I and J imply

$$X \cap (I \cap J) = (X \cap I) \cap J \subseteq (Y \cap I) \cap J = Y \cap (I \cap J)$$

and

$$Y \cap (I \cap J) = (Y \cap J) \cap I \subseteq (X \cap J) \cap I = X \cap (I \cap J),$$

i.e.,

$$X \cap (I \cap J) = Y \cap (I \cap J).$$

The sets X and Y are asymptotically equal because $I \cap J$ is a positive, unbounded interval.

***O*, Ω, and Θ.** Asymptotic notation (O, Ω, Θ) is defined herein only for certain real-valued functions defined on certain sets of real numbers. Unlike many other sources (e.g., [Kn]), we follow the convention of [CLRS] that requires asymptotically related functions to be *asymptotically non-negative*, i.e., they must have only non-negative values for sufficiently large elements of their domains. (*Asymptotically positive* functions are defined similarly.) Apart from an exception ($\Theta(1)$) described later, we require asymptotically related functions to have asymptotically related domains that are unbounded above. We do not require equality or asymptotic equality of domains. However, each of the relations $f = O(g)$, $f = \Omega(g)$, and $f = \Theta(g)$ require the domain of f to be asymptotically contained in the domain of g. (The relation is viewed as an estimate for f, not an estimate for g. Asymptotic containment is more convenient than asymptotic equality in some instances. A cumbersome restriction of the function on the right-hand side is sometimes avoided.) Apart from the aforementioned exception of $\Theta(1)$, the relation $f = \Theta(g)$ means $sup\ domain(f) = \infty$ and there exists a positive, unbounded interval H and $a, b \in \boldsymbol{R}^+$ such that

$$domain(f) \cap H \subseteq domain(g)$$

and

$$0 \leq ag(x) \leq f(x) \leq bg(x)$$

for all $x \in domain(f) \cap H$. The relation $f = \Omega(g)$ is satisfied if $sup\ domain(f) = \infty$ and there exists a positive, unbounded interval I and $c \in \boldsymbol{R}^+$ such that

$$domain(f) \cap I \subseteq domain(g)$$

and

$$0 \leq cg(x) \leq f(x)$$

for all $x \in domain(f) \cap I$. The relation $f = O(g)$ is satisfied if $sup\ domain(f) = \infty$ and there exists a positive, unbounded interval J and $d \in \boldsymbol{R}^+$ such that

$$domain(f) \cap J \subseteq domain(g)$$

and

$$0 \leq f(x) \leq dg(x)$$

for all $x \in domain(f) \cap J$. The relation $f = \Theta(g)$ is equivalent to the combination of $f = O(g)$ and $f = \Omega(g)$ since the intersection of two positive, unbounded intervals is a positive, unbounded interval.

$\boldsymbol{\Theta(1)}$ on set with finite upper bound. Although the standard definition of Big-Theta asymptotic notation is not meaningful for the restriction of a real-valued function to a set of real numbers with a finite upper bound, language such as

$$T(x) = \Theta(1) \text{ for } x \in [1, x_0]$$

appears in [Le]. The apparent intended meaning is that $T([1, x_0])$ has a positive lower bound and a finite upper bound. Those are the conditions required by the applications in [Le], namely the base cases of the inductions in the proofs of Theorems 1 and 2. [CLRS] uses $\Theta(1)$ similarly, e.g., "… $T(n) = \Theta(1)$ for sufficiently small n …" (p. 67).

We adopt the same convention. Suppose X is a set of real numbers with a finite upper bound. A function f on X satisfies $f = \Theta(1)$ if f is real valued and has a positive lower bound and a finite upper bound. We also say f is $\Theta(1)$. If F is a function whose domain contains a set X, phrases equivalent to $F|_X = \Theta(1)$ include "$F(x) = \Theta(1)$ for $x \in X$" and "F is $\Theta(1)$ on X". According to our definition, every function is $\Theta(1)$ on the empty set.

Locally $\boldsymbol{\Theta(1)}$ functions. A real-valued function T on a set D of real numbers is *locally $\Theta(1)$* if $T|_S = \Theta(1)$ for every bounded subset S of D. The reader might reasonably expect $T: D \to \boldsymbol{R}$ to be called *locally $\Theta(1)$* if for all $x \in D$ there exists an open set W with $x \in W$ and $T|_{D\cap W} = \Theta(1)$. The difference is illustrated by the function $1/x$ on $(0, \infty)$, which is $\Theta(1)$ on the neighborhood $(u/2\,, 2u)$ of u for all $u \in (0, \infty)$. It is not $\Theta(1)$ on $(0,1)$ and is therefore not locally $\Theta(1)$. The author apologizes for any confusion caused by our more restrictive definition.

$\boldsymbol{\Theta(1)}$ on a union. Suppose T is a real valued function on a set D of real numbers such that $T|_X = \Theta(1)$ and $T|_Y = \Theta(1)$ for some subsets X and Y of D. We claim that $T|_{X\cup Y} = \Theta(1)$. There exist $a, b, c, d \in \boldsymbol{R}^+$ and $u, w \in [-\infty, \infty)$ such that $a \le T(x) \le b$ for all $x \in X \cap (u, \infty)$ and $c \le T(y) \le d$ for all $y \in Y \cap (w, \infty)$. Furthermore, we may let $u = -\infty$ if $\sup X < \infty$, and $w = -\infty$ if $\sup Y < \infty$. (Our definition of $\Theta(1)$ depends on whether the underlying set has a finite upper bound.) Define positive real numbers $L = \min\{a, c\}$ and $U = \max\{b, d\}$, so $L \le T(z) \le U$ for all

$$z \in (X \cup Y) \cap (\max(u, w)\,, \infty).$$

If $\sup(X \cup Y) = \infty$, then $T|_{X\cup Y} = \Theta(1)$ because $\max(u, w) < \infty$. If $\sup(X \cup Y) < \infty$, then $\sup X < \infty$ and $\sup Y < \infty$, which implies

$$(X \cup Y) \cap (\max(u, w)\,, \infty) = (X \cup Y) \cap (-\infty, \infty) = X \cup Y$$

and $T|_{X\cup Y} = \Theta(1)$. This principle shall be applied henceforth without further comment.

Arithmetic on $\boldsymbol{[0, \infty]}$. Measure theory [Ta] employs an extension of addition and multiplication from $\boldsymbol{R}^+$ to the extended non-negative real axis $[0, \infty]$ that defines

$$x + \infty = \infty + x = \infty$$

for all $x \in [0, \infty]$,

$$y \cdot \infty = \infty \cdot y = \infty$$

for all non-zero $y \in (0, \infty]$, and (for upward continuity of multiplication)

$$0 \cdot \infty = \infty \cdot 0 = 0.$$

The resulting operations are commutative and associative, with multiplication distributive over addition. They also preserve non-strict inequalities: if $a, b, c, d \in [0, \infty]$ such that $a \le b$ and $c \le d$, then

$$a + c \le b + c \le b + d$$

and

$$ac \leq bc \leq bd.$$

Cancellation does not work with addition or multiplication by ∞ (or multiplication by 0). For example,

$$3 + \infty = 4 + \infty = \infty,$$

$$3 \cdot \infty = 4 \cdot \infty = \infty,$$

and

$$3 \cdot 0 = 4 \cdot 0 = 0,$$

but $3 \neq 4$. For this reason among others, the quotients ∞/∞ and $0/0$ are undefined as is the difference $\infty - \infty$.

We use a little arithmetic on $[0, \infty]$ as described above, although we don't need the product of 0 and ∞. Strictly as convenient notational shorthand, we also adopt the nonstandard conventions that $x/0 = \infty$ for all non-zero $x \in (0, \infty]$ and $y/\infty = 0$ for all finite $y \in [0, \infty)$, but make very limited use of these two definitions: The statements of Lemmas 2.8 and 2.10(2) are simplified by the first convention, and the language of Lemmas 2.9(4) and 2.10(4) are simplified by the second. Our conventions for division by 0 and ∞ do not affect our applications of the aforementioned propositions; the applications do not involve either 0 or ∞ as a denominator.

The indeterminate forms $0/0$ and ∞/∞ are the only undefined fractions with numerators and denominators in $[0, \infty]$. Our definition of arithmetic on the extended non-negative real axis satisfies

$$\frac{a}{b} = a \cdot \frac{1}{b}.$$

for all $a, b \in [0, \infty]$ with a/b defined, i.e., except when $a = b = 0$ or $a = b = \infty$.

Our conventions imply continuity of division provided we avoid the indeterminate forms $0/0$ and ∞/∞: If a sequence $x_n \in [0, \infty]$ approaches $X \in [0, \infty]$ and a sequence $y_n \in [0, \infty]$ approaches $Y \in [0, \infty]$ such that x_n/y_n and X/Y are defined, then x_n/y_n approaches X/Y.

Division by 0 and ∞ is a notational convenience only and does not have all the usual properties of division. For example,

$$0 \cdot (1/0) = 0 \cdot \infty = 0 \neq 1,$$

$$\infty \cdot (1/\infty) = \infty \cdot 0 = 0 \neq 1,$$

and

$$\frac{1}{0 \cdot \infty} = \frac{1}{0} = \infty \neq 0 = \infty \cdot 0 = \frac{1}{0} \cdot \frac{1}{\infty}.$$

Under suitable circumstances, the denominator of a product of fractions is the product of denominators: If $x, y \in [0, \infty]$ such that $\{x, y\} \neq \{0, \infty\}$, then

$$\frac{1}{xy} = \frac{1}{x} \cdot \frac{1}{y}.$$

The familiar identity

$$\frac{x}{y} = \frac{1}{y/x}$$

of $\boldsymbol{R}^+$ can also be partially extended to $[0, \infty]$. Observe that

$$\frac{a}{\infty} = 0 = \frac{1}{\infty} = \frac{1}{\infty/a}$$

for all finite $a \in [0, \infty)$, and

$$\frac{b}{0} = \infty = \frac{1}{0} = \frac{1}{0/b},$$

for all non-zero $b \in (0, \infty]$, so

$$\frac{x}{y} = \frac{1}{y/x}$$

for all $x, y \in [0, \infty]$ for which x/y (equivalently y/x) is defined, i.e., $\{x, y\} \notin \{\{0\}, \{\infty\}\}$.

Division is well behaved with respect to inequalities. If $a, b, c, d \in [0, \infty]$ satisfy the inequalities $a \leq b$ and $c \leq d$ such that a/d and b/c are defined, i.e.,

$$\{a, d\}, \{b, c\} \notin \{\{0\}, \{\infty\}\},$$

we claim that

$$\frac{a}{d} \leq \frac{b}{c}.$$

If $a, b, c, d \in \boldsymbol{R}^+$, the inequality above is proved by dividing the relation $ac \leq bd$ by cd. Thus, we may assume one or more of a, b, c, d is 0 or ∞, which implies either $0 \in \{a, c\}$ or $\infty \in \{b, d\}$. Then either $a/d = 0$ or $b/c = \infty$, so the claimed inequality is satisfied.

Set notation. We distinguish between and $S \subseteq T$ and $S \subset T$. The former means S is a subset of T, i.e., every element of S is an element of T. The latter means S is a proper subset of T, i.e., $S \subseteq T$ and $S \neq T$. Of course, $\emptyset$ represents the empty set.

We inconsistently use both $A - B$ and $A \setminus B$ to denote the difference of sets A and B.

Upper and lower subsets. An *initial subset* of a set S of real numbers is a subset L of S that satisfies $x < y$ for all $x \in L$ and all $y \in S - L$. An initial subset is also called a *lower subset*. An *upper subset* of S is a subset U of S that satisfies $u > w$ for all $u \in U$ and all $w \in S - U$. (Of course, these notions can be defined more generally for partially ordered sets; however, we require them only for sets of real numbers).

The complement of a lower subset is an upper subset, and vice versa. For example, the lower subsets of the positive integers are

$$\phi, \{1\}, \{1,2\}, \{1,2,3\}, \dots, \boldsymbol{Z}^{+},$$

and the upper subsets are

$$\boldsymbol{Z}^{+}, \{2,3,4,\dots\}, \{3,4,5,\dots\}, \{4,5,6\dots\}, \dots, \phi.$$

Lower subsets of $[1,\infty)$ include $[1,2]$ and $[1,2)$, with corresponding upper subsets $(2,\infty)$ and $[2,\infty)$, respectively.

If X is a subset of a set S of real numbers, the *minimum initial subset of S containing X* is the intersection M of all initial subsets of S containing X, i.e.,

$$M = \big(S \cap (-\infty, \sup X)\big) \cup (X \cap \{\sup X\}).$$

Functions. There are two common conventions for the definition of a function, with each having advantages and disadvantages. Since they have some incompatibilities, we shall examine our choice of definition in detail.

A *binary relation* is a set of ordered pairs. Given a binary relation R,

$$\{x : (x,y) \in R \text{ for some } y\}$$

and

$$\{y : (x,y) \in R \text{ for some } x\}$$

are sets ([Je] p. 10) called the *domain* and *range*, respectively. We sometimes refer to them as $domain(R)$ and $range(R)$. A *functional graph* is a binary relation with no two elements having the same first component.

We define a *function* to be a functional graph. Functions are sometimes called *maps*, *mappings,* or *transformations* among other names. Of course, given a function f and $x \in domain(f)$, the expression $f(x)$ represents the unique $y \in range(f)$ for which $(x,y) \in f$. The element y is called the *value of f at x* or *the image of x* (*under f*). We also say f *maps* x to y. The mapping of x to y is sometimes denoted by $x \mapsto y$. The graph of a function f is the set

$$\big\{\big(x, f(x)\big) : x \in domain(f)\big\}.$$

By our definition, a function is equal to its own graph. The simplest and least interesting example of a function is the *empty function*, which has an empty domain, range, and graph. All other functions are *non-empty*.

If f is a function with domain A and the range of f is contained in a set B, we call f a function *from A to* (or *into*) B and say f *maps A to* (or *into*) B. The notation $f: A \to B$

has the same meaning. We adopt the nonstandard terminology that B is *a codomain* of f. In function language, *target* is a synonym for *codomain*.

Notice our reference to *a* codomain, not *the* codomain. According to our nonstandard definition, every function has an infinite number of codomains. Functions f and g are equal if and only if they have the same domains and $f(x) = g(x)$ for all x in their common domain.

Many published definitions of a function specify a unique codomain (or target). For example, Bourbaki ([Bo], p. 81) defines a function to be an ordered triple $f = (F, A, B)$, where F is a functional graph, A is what we call the domain of F, and B is what we call a codomain (or target) of F. Bourbaki considers A and B to be the domain and the unique target, respectively, of f (not F). They call f a *function from A to B* and use the notation $f: A \to B$. Suppose $C \neq B$ is another set that contains $f(x)$ for all $x \in A$. According to Bourbaki's definition, f is not a function from A to C and the notation $f: A \to C$ is inapplicable. In their framework, functions f and g are equal if and only if they have the same graph and the same target. Equality of domains follows from equality of graphs.

Consider the constant function $\alpha: \boldsymbol{Z} \to \boldsymbol{R}$ defined by $\alpha(n) = \pi$. The function α maps $\boldsymbol{Z}$ to $\boldsymbol{R}$. According to our convention, α also maps $\boldsymbol{Z}$ to $\{\pi\}$, i.e., $\alpha: \boldsymbol{Z} \to \{\pi\}$. Under the Bourbaki definition of a function,

$$\alpha = (\{(n, \pi) : n \in \boldsymbol{Z}\}, \boldsymbol{Z}, \boldsymbol{R})$$

and

$$\beta = (\{(n, \pi) : n \in \boldsymbol{Z}\}, \boldsymbol{Z}, \{\pi\})$$

are distinct functions although $domain(\alpha) = \boldsymbol{Z} = domain(\beta)$ and $\alpha(n) = \beta(n)$ for all $n \in \boldsymbol{Z}$. Futhermore, the set $\boldsymbol{R}$ is the only target of α, and the notation $\alpha: \boldsymbol{Z} \to \{\pi\}$ is inapplicable. Similarly, the set $\{\pi\}$ is the only target of β, and the notation $\beta: \boldsymbol{Z} \to \boldsymbol{R}$ is inapplicable.

Incorporation of a unique codomain into the definition of a function is harmonious with category theory—see [Ma]. An earlier version of this document implicitly assumed such a definition. However, there was some bureaucratic overhead. At several places in later sections, two functions were required to differ only in their codomains. A reader unused to such gyrations might find them confusing, so the author reluctantly switched definitions. Hopefully, our convention does not confuse readers (such as myself) who prefer a unique codomain for each function.

Given a function $f: A \to B$ and a subset S of A, the *image of S under f* is the set

$$f(S) = \{f(x) : x \in S\}.$$

Our use of $f(S)$ is an abuse of notation; there is a potential ambiguity if S is also an element of A; however, the meaning should always be clear from context. Some authors write $f[S]$ instead of $f(S)$ to distinguish the image of a subset of the domain from the function's value at an element of the domain. If $Y = f(S)$, we say f maps S *onto* Y. The image $f(A)$ of the domain A under f is also called the *image of* f.

According to our definitions, the image and range of f are identical. Some authors (especially in earlier times) define a function's range differently. Their *range* is what we would call a codomain or target. (Some such authors require each function to have exactly one range; some allow multiple ranges per function; others are silent or ambiguous on the subject.)

A function f is *injective* (AKA *1-1* or one-to-one) if $x = y$ for all $x, y \in domain(f)$ that satisfy $f(x) = f(y)$. An *injection* is an injective function. If functions are defined to have unique codomains, then there are meaningful definitions of *surjective*, *surjection*, *bijective*, and *bijection*: A function is *surjective* (AKA *onto*) if the image and codomain of the function are the same sets. A *surjection* is a surjective function. A function is *bijective* if it is injective and surjective. A *bijection* is a bijective function and is also called a *one-to-one correspondence*. However, functions have multiple codomains according to our definition of a function, so the terms *surjective*, *surjection*, *bijective*, and *bijection* are not well defined. Nonetheless, we say a function maps its domain *onto* its image.

Mathematicians are sometimes inconsistent in their language about functions. For example, Bourbaki ([Bo] pp. 81–82) gives the ordered triple definition of a function, then says: "Throughout this series we shall often use the word "function" in place of "functional graph"." Moschovakis ([Mo], pp. 3–4, 38–40) uses a definition of function that is equivalent to ours but also defines surjections and bijections. Like some other authors, he uses context and double-headed arrows to indicate specific ranges (his *range* is our non-unique *codomain*) for surjections and only refers to surjections or bijections when the relevant range is identified by context and notation. We also reserve the right to abuse terminology by using the terms *surjective*, *surjection*, *bijective*, and *bijection* when a particular choice of codomain is apparent from context.

Our convention for composition of functions is dictated by our choice of definition of a function: If $g: W \to X$ and $f: X \to Y$ are functions, then the composition of f and g is the function

$$f \circ g: W \to Y$$

defined by

$$(f \circ g)(x) = f\big(g(x)\big)$$

for all $x \in$ W. Some authors who require unique targets for functions also require the domain of f to be *the* codomain of g. We require only that the domain of f contains the range of g.

Given $f: A \to B$ and a set T, the *preimage* (or *inverse image*) *of* T *under* f is the set

$$f^{-1}(T) = \{a \in A : f(a) \in T\}.$$

This notation can cause some confusion since preimages are defined regardless of whether the function f has an inverse. Some authors use alternative notation such as $f_{-1}(T)$ to avoid confusion. The notation $f^{-1}[T]$ also appears in the literature. The set T need not be contained in the range of f. The set T is commonly assumed to be contained in the codomain; however, in our language, every set is contained in some codomain of the function.

The phrase "f is a function on A" means that f is a function with domain A. The identity map on a set A is the function $id: A \to A$ defined by $id(a) = a$ for all $a \in A$.

Restriction and extension of functions are dual concepts: If f is a function on a set A and S is a subset of A, then the *restriction* of f to S is the function $f|_S: S \to f(S)$ defined by $f|_S(x) = f(x)$ for all $x \in S$. If A^* is a set containing A, an *extension of* f *to* A^* is a function f^* on A^* such that $f^*|_A = f$. We also say f^* *extends* f *to* A^*.

If f and g are functions, and

$$S \subseteq domain(f) \cap domain(g),$$

then f *agrees with* g *on* S if $f|_S = g|_S$, i.e., $f(x) = g(x)$ for all $x \in S$.

In some contexts, a function exponent represents exponentiation of function values. For example, $\sin^2 x = (\sin x)^2$ and $\log^n(x) = \left(\log(x)\right)^n$. However, composition of functions is sometimes intended. If $f: S \to S$ is a function from a set S to itself, we may let f^0 be the identity function on S and recursively define $f^n: S \to S$ by $f^n = f \circ f^{n-1}$ for each positive integer n. Suppose $f: S \to S$ is a bijection, so the function f has an inverse f^{-1}: S → S. For each negative integer k, we define $f^k: S \to S$ by

$$f^k = (f^{-1})^{|k|},$$

so

$$f^k = \left(f^{|k|}\right)^{-1}.$$

The meaning of a function exponent is usually clear from context. However, we typically identify those instances where a function exponent refers to composition of functions.

2. Polynomial Growth

This section defines *polynomial-growth functions* and delineates their basic properties. All nontrivial polynomial-growth functions are shown to be positive, with reciprocals that are also polynomial-growth functions. Behavior on lower subsets of their domains is analyzed in detail. A connection between polynomials and polynomial growth is made explicit. We show that some positive polynomial functions do not have polynomial growth. Conditions for preservation of polynomial growth under Θ-equivalence are determined. We also provide a convenient interpretation of polynomial growth based on bounded dynamic ranges. There are no surprises.

Polynomial growth is an unsatisfactory description for the class of functions in question. Some variant of "uniformly bounded dynamic range" might be preferable. However, we defer to Leighton [Le] in his choice of this terminology.

Leighton defines a polynomial-growth condition in the context of a recurrence involving a non-negative real-valued function g and coefficients $b_1, \dots, b_k$, where $0 < b_i < 1$ for each index i. (There are also coefficients $a_1, \dots, a_k > 0$.) The domain of g is never specified, but Leighton's polynomial-growth condition implicitly requires the domain to contain $[\min S, \infty)$ where $S = \{b_1, \dots, b_k\}$.

Definition. Let S be a non-empty, finite subset of the open interval $(0,1)$. A *candidate* (for Leighton's polynomial-growth condition) relative to S is a non-negative real-valued function whose domain contains $[\min S, \infty)$.

Leighton's polynomial-growth condition is a property of the restriction of g to $[\min S, \infty)$:

Definition. Let S be a non-empty, finite subset of the open interval $(0,1)$. A function g satisfies Leighton's polynomial-growth condition relative to S if g is a candidate relative to S, and there exist positive real numbers c_1 and c_2 such that for all $\beta \in S$ and all $x \geq 1$,

$$c_1 g(x) \leq g(u) \leq c_2 g(x)$$

for all $u \in [\beta x, x]$.

For convenience, we sometimes ignore the choice of S in the two preceding definitions. When we say a function satisfies or is a candidate for Leighton's polynomial-growth condition without reference to such a set, the existence of an appropriate subset S of $(0,1)$ is implied.

The ad hoc definition of Leighton's polynomial-growth condition is artificially dependent on the choice of S. We define a *polynomial-growth function* in a slightly different fashion and provide a simple characterization (Lemma 2.16) of such functions when their domains are intervals. Corollary 2.17 says that a candidate relative to S satisfies Leighton's polynomial-growth condition relative to S if and only if its restriction to $[\min S, \infty)$ satisfies our definition of polynomial growth. We note that [Le] makes no use of his function g's behavior outside the interval $[\min S, \infty)$. Our definition will be derived from the following simple variation of Leighton's condition:

Definition. Let $b > 1$. A *b-polynomial-growth function* is a non-negative real-valued function g on a positive, unbounded interval I such that there exist $c_1 > 0$ and $c_2 > 0$ with

$$c_1 g(x) \leq g(u) \leq c_2 g(x)$$

for all $x \in I$ and all $u \in [x, bx]$.

A b-polynomial-*growth function* is said to satisfy the *b-polynomial-growth condition*. Lemma 2.16 will show that satisfaction of the b-polynomial-growth condition does not depend on the choice of b.

Since the domain of a b-polynomial-growth function is a positive, unbounded interval, the domain contains $[x, bx]$ for all x in the domain as implicitly asserted in the definition above. Indeed, the domain contains $[x, \infty)$.

Lemma 2.1. A candidate g relative to a non-empty, finite subset S of $(0,1)$ satisfies Leighton's polynomial-growth condition relative to S if and only if the restriction of g to $[\min S, \infty)$ is a b-polynomial-growth function where $b = 1/\min S$.

Proof. It follows from $\min S \in (0,1)$ that $b > 1$. Suppose the restriction of g to $[\min S, \infty)$ is a b-polynomial-growth function. By definition of a b-polynomial-growth function, there exist positive real numbers d_1 and d_2 such that

$$d_1 g(t) \leq g(w) \leq d_2 g(t)$$

for all $t \in [\min S, \infty)$ and all $w \in [t, bt]$. Let $\beta \in S$ and $x \geq 1$, so $\beta \geq 1/b$ and

$$\min S = \frac{1}{b} \leq \frac{x}{b} \leq \beta x < x.$$

Thus, $x/b \in [\min S, \infty)$ and $[\beta x, x] \subseteq [x/b\,, x]$. Let $u \in [\beta x, x]$, so $u, x \in [x/b\,, x]$ and

$$\frac{d_1}{d_2}g(x) \le d_1 g(x/b) \le g(u) \le d_2 g(x/b) \le \frac{d_2}{d_1}g(x).$$

It follows from $d_1 > 0$ and $d_2 > 0$ that $d_1/d_2 > 0$ and $d_2/d_1 > 0$. Therefore, g satisfies Leighton's polynomial-growth condition relative to S with $c_1 = d_1/d_2$ and $c_2 = d_2/d_1$.

We now prove the converse. Suppose g satisfies Leighton's polynomial-growth condition relative to S. Let c_1 and c_2 be as in the definition of that condition. Suppose $x \in [\min S, \infty) = [1/b, \infty)$, so $bx \ge 1$. Let $u \in [x, bx]$. Since $1/b \in S$ and $x \in [x, bx]$,

$$\frac{c_1}{c_2}g(x) \le c_1 g(bx) \le g(u) \le c_2 g(bx) \le \frac{c_2}{c_1}g(x).$$

It follows from $c_1 > 0$ and $c_2 > 0$ that $c_1/c_2 > 0$ and $c_2/c_1 > 0$. Thus, the restriction of g to $[\min S, \infty)$ is a b-polynomial-growth function. □

The definition of a b-polynomial-growth function requires the domain to be a positive, unbounded interval. We are also interested in behavior of functions with other positive domains such as sets of positive integers:

Definition. A function has *polynomial growth* if it can be extended to a b-polynomial-growth function for some $b > 1$.

A function with polynomial growth is called a *polynomial-growth function* and is said to satisfy the *polynomial-growth condition*. The phrase *polynomial-growth condition* does not refer to Leighton's polynomial-growth condition unless explicitly identified as such. Lemma 2.16 will establish independence of the polynomial-growth condition from the choice of b.

If a function's restriction to a subset X of its domain has polynomial growth, we say the function has polynomial growth on X or satisfies the polynomial-growth condition on X.

Our definition of a polynomial-growth function is intentionally simple. However, the definition is arguably too simple because it is satisfied by the empty function. Inclusion of the empty function reduces verbiage in some contexts but is a minor nuisance in others. Some later definitions are similarly permissive with respect to the empty set. The choice between inclusion and exclusion of these vacuous cases has no significance.

We catalog some immediate consequences of our definitions:

Lemma 2.2.

(1) All polynomial-growth functions are real-valued and non-negative with positive domains.

(2) The restriction of a polynomial-growth function to a subset of its domain has polynomial growth.

(3) Let $b > 1$. The restriction of a b-polynomial-growth function to an unbounded interval in its domain is also a b-polynomial-growth function.

(4) A function on a positive, unbounded interval has polynomial growth if and only if it is a b-polynomial-growth function for some $b > 1$.

(5) A function has polynomial growth if and only if it can be extended to a polynomial-growth function on some positive, unbounded interval.

(6) A function f with a non-empty positive domain D has polynomial growth if and only if f can be extended to a polynomial-growth function on the minimum positive, unbounded interval containing D.

(7) A function f is a candidate for Leighton's polynomial-growth condition if and only if f is real-valued and non-negative with a domain that contains a positive, unbounded interval that properly contains $[1, \infty)$.

(8) If a function f satisfies Leighton's polynomial-growth condition, then f has polynomial growth on $[1, \infty)$.

Proof. (1) A polynomial-growth function is real-valued and non-negative with a positive domain because it is the restriction of some b-polynomial-growth function, which by definition is real-valued and non-negative with a positive domain.

(2) If g is a polynomial-growth function, there exists $b > 1$ and a b-polynomial-growth function h that is an extension of g. If S is a subset of the domain D of g, then

$$g|_S = (h|_D)|_S = h|_S,$$

i.e., the function h is a b-polynomial-growth extension of $g|_S$. By definition, $g|_S$ has polynomial growth.

(3) Let h be a b-polynomial-growth function. Its domain is a positive, unbounded interval J. Suppose I is an unbounded interval contained in J. The interval I is positive because J is positive. Let $g = h|_I$. The function g is non-negative because h is non-negative. By definition of a b-polynomial-growth function, there exist positive real numbers c_1 and c_2 such that

$$c_1 h(y) \leq h(v) \leq c_2 h(y)$$

for all $y \in J$ and all $v \in [y, by]$. Suppose $x \in I$ and $u \in [x, bx]$. Since I is a positive, unbounded interval, we have $[x, bx] \subset I \subseteq J$. Then

$$c_1 g(x) = c_1 h(x) \leq h(u) = g(u) = h(u) \leq c_2 h(x) = c_2 g(x).$$

In particular,

$$c_1 g(x) \leq g(u) \leq c_2 g(x).$$

Thus g satisfies all the requirements of a b-polynomial-growth function.

(4) Let g be a function on a positive, unbounded interval. If g satisfies the b-polynomial-growth condition for some $b > 1$, then g is a (trivial) b-polynomial-growth extension of itself, which implies g has polynomial growth. Conversely, let h be a polynomial-growth function on a positive, unbounded interval I. Then h has a b-polynomial-growth extension H for some $b > 1$. Since $h = H|_I$, part (3) implies h is also a b-polynomial-growth function.

(5) follows from (4) and the definition of a b-polynomial-growth function.

(6) Let I be the minimum positive, unbounded interval containing D. (There is no such minimum if D is empty.) If f can be extended to a polynomial-growth function on I, then f has polynomial growth by part (5). Conversely, suppose f has polynomial growth, so (5) implies f can be extended to a polynomial-growth function h on some positive, unbounded interval J, which contains I. The function $g = h|_I$ has polynomial growth by (2). Furthermore,

$$f = h|_D = (h|_I)|_D = g|_D,$$

i.e., g is an extension of f to I.

(7) By definition, each candidate for Leighton's polynomial-growth condition is a non-negative real-valued function with a domain properly containing $[1, \infty)$. Conversely, suppose f is a non-negative real-valued function, and $domain(f)$ contains a positive unbounded interval I that properly contains $[1, \infty)$, so there exists $b \in I \cap (0,1)$. Then $[b, \infty) \subseteq I \subseteq domain(f)$, which implies f is a candidate for Leighton's polynomial-growth condition relative to $\{b\}$.

(8) There exists a non-empty, finite subset S of $(0,1)$ such that f satisfies Leighton's polynomial-growth condition relative to S. In particular, f is a candidate relative to S, so the domain of f contains $[\min S, \infty)$. Let g be the restriction of f to $[\min S, \infty)$. Lemma 2.1 implies g is a b-polynomial-growth function where $b = 1/\min S$. The interval $[\min S, \infty)$ contains $[1, \infty)$. Let h be the restriction of g to $[1, \infty)$. The function h satisfies our definition of polynomial growth because g is a b-polynomial-growth extension of h. The function f has polynomial growth on $[1, \infty)$ because

$$f|_{[1,\infty)} = \left(f|_{[\min S,\infty)}\right)|_{[1,\infty)} = g|_{[1,\infty)} = h.$$

□

Polynomial-growth functions on intervals need not be differentiable or even continuous. For example, let g be the nowhere continuous function with domain $(0,\infty)$ defined by $g(x) = 1$ for rational x and $g(x) = 2$ for irrational x. The function g also fails to be Riemann integrable on any non-degenerate, compact interval in its domain. For each $b > 1$, the function g satisfies the definition of a b-polynomial-growth function with $c_1 = 1/2$ and $c_2 = 2$. By Lemma 2.2(4), g has polynomial growth. Lemmas 2.1 and 2.2(3) imply the function g also satisfies Leighton's polynomial-growth condition relative to S whenever $\phi \neq S \subseteq (0,1)$ is finite. However, [Le] contains integrals involving functions whose only explicit requirement is satisfaction of Leighton's polynomial-growth condition, which does not imply local Riemann integrability.

We now consider the simplest class of polynomial-growth functions (which vacuously includes the empty function):

Lemma 2.3. Non-negative constant functions on positive sets have polynomial growth. Non-negative constant functions on positive, unbounded intervals are b-polynomial-growth functions for all $b > 1$.

Proof. Non-negative constant functions on positive, unbounded intervals satisfy the definition of a b-polynomial-growth function for all $b > 1$ with $c_1 = c_2 = 1$. Non-negative constant functions on positive sets can be extended to non-negative constant functions on $(0,\infty)$ and therefore have polynomial growth. □

As a special case of Lemma 2.3, all identically zero functions on positive sets have polynomial growth. We later show (Lemma 2.7) that a polynomial-growth function is either positive or identically zero. (Only the empty function is both.) The obvious proposition below is a first step in that direction:

Lemma 2.4. If g is a b-polynomial-growth function for some $b > 1$, and x is an element of the domain of g, then the restriction of g to $[x, bx]$ is either positive or identically zero.

Proof. By definition of a b-polynomial-growth function, g is non-negative and there exist positive real numbers c_1 and c_2 such that

$$c_1 g(x) \leq g(u) \leq c_2 g(x)$$

for all $u \in [x, bx]$. If $g(x) = 0$, then

$$0 = c_1 \cdot 0 \leq g(u) \leq c_2 \cdot 0 = 0$$

for all $u \in [x, bx]$, i.e., the restriction of g to $[x, bx]$ is identically zero. If $g(x) \neq 0$, the non-negativity of g implies $g(x) > 0$, so

$$g(u) \geq c_1 g(x) > 0$$

for all $u \in [x, bx]$, i.e., the restriction of g to $[x, bx]$ is positive. □

Lemma 2.5. Suppose g is a b-polynomial-growth function for some $b > 1$, and x is an element of the domain of g. For each positive integer n, the restriction of g to $[x, b^n x]$ is either positive or identically zero.

Proof. By Lemma 2.4, the restriction of g to $[x, b^1 x]$ is either positive or identically zero. Let n be any positive integer for which the restriction of g to $[x, b^n x]$ is either positive or identically zero. Lemma 2.4 implies the restriction of g to $[b^n x, b^{n+1} x]$ is either positive or identically zero. Since

$$[x, b^{n+1} x] = [x, b^n x] \cup [b^n x, b^{n+1} x]$$

and

$$b^n x \in [x, b^n x] \cap [b^n x, b^{n+1} x],$$

the restriction of g to $[x, b^{n+1} x]$ is positive if $g(b^n x) > 0$. Similarly, the restriction of g to $[x, b^{n+1} x]$ is identically zero if $g(b^n x) = 0$. The lemma follows by induction. □

Lemma 2.6. If g is a b-polynomial-growth function for some $b > 1$, and x is an element of the domain of g, then the restriction of g to $[x, \infty)$ is either positive or identically zero.

Proof. All b-polynomial-growth functions are non-negative. In particular, $g(x) \geq 0$. Lemma 2.5 implies the restriction of g to $[x, b^n x]$ is positive for all $n \in \boldsymbol{Z}^+$ if $g(x) > 0$, and each such restriction is identically zero if $g(x) = 0$. The proposition follows from

$$[x, \infty) = \bigcup_{n=1}^{\infty} [x, b^n x].$$

□

Lemma 2.7. A polynomial-growth function is either positive or identically zero.

Proof. A polynomial-growth function g can be extended to a b-polynomial-growth function h for some $b > 1$. Let I be the domain of h. By definition of a b-polynomial-growth function, h is non-negative and I is a positive, unbounded interval. Suppose h is not positive. Then h has a root z. Let y be any element of I, and define $x = \min(y, z)$, so that $x \in I$. Since $z \in [x, \infty)$, Lemma 2.6 implies the restriction of h to $[x, \infty)$ is identically zero. In particular, $h(y) = 0$. Therefore, h is identically zero if h is not positive. Since h is an extension of g, we conclude that g is either positive or identically zero. □

Example of non-negative polynomial function without polynomial growth. The function $g(x) = (x-1)^2$ on $(0,\infty)$ has 1 as its unique root, so Lemma 2.7 implies g does not have polynomial growth on any positive set properly containing $\{1\}$.

Definition. For each positive set S, define

$$\text{Ratios}(S) = \left\{\frac{x}{y} : x, y \in S\right\},$$

and

$$\Lambda(S) = \sup \text{Ratios}(S).$$

For each positive function g and each subset X of its domain, define

$$\Lambda_g(X) = \Lambda\big(g(X)\big)$$

where $g(X) = \{g(t) : t \in X\}$. The quantity $\Lambda(S)$ is called the *dynamic range* of S, and $\Lambda_g(X)$ is the dynamic range of g on X. If X is the domain of g, then $\Lambda_g(X)$ is the dynamic range of g.

The lemma below provides an arguably more natural interpretation of dynamic ranges, although the definition above is more convenient in some contexts. The proposition's statement uses our nonstandard convention that $x/0 = \infty$ for all non-zero $x \in (0,\infty]$.

Lemma 2.8. If S is a non-empty positive set, then

$$\Lambda(S) = \frac{\sup S}{\inf S}.$$

Proof. Since $\sup S > 0$ and $0 \le \inf S < \infty$, the fraction $\sup S/\inf S$ is defined. If $x, y \in S$, then $x/y \le \sup S/\inf S$. Therefore, $\Lambda(S) \le \sup S/\inf S$. There exists infinite sequences x_n and y_n in S such that x_n approaches $\sup S$ and y_n approaches $\inf S$. (The sequences may have repeated terms.) Then x_n/y_n approaches $\sup S/\inf S$. Therefore, $\sup S/\inf S \le \Lambda(S)$. □

$\Lambda(\emptyset)$ trivialities. Observe that

$$\Lambda(\emptyset) = \sup \text{Ratios}(\emptyset) = \sup \emptyset = -\infty.$$

For each positive function g, we have

$$\Lambda_g(\emptyset) = \Lambda\big(g(\emptyset)\big) = \Lambda(\emptyset) = -\infty.$$

The conclusion of Lemma 2.8 is invalid when S is the empty set because $\Lambda(\emptyset) = -\infty$, whereas $(\sup \emptyset)/(\inf \emptyset)$ is the undefined ratio $(-\infty)/\infty$.

The next lemma lists some of the most obvious properties of dynamic ranges. The quotient in (4) uses our nonstandard convention that $x/\infty = 0$ for all finite $x \in [0, \infty)$.

Lemma 2.9. If S is a positive set, then

(1) $\Lambda(S) \geq 1$ if S is non-empty.

(2) $\Lambda(S) < \infty$ if and only if $\inf S > 0$ and $\sup S < \infty$.

(3) $\Lambda(S) < \infty$ if S is finite.

(4) If x is an element of S, then $x/\Lambda(S) \leq \inf S$ and $\Lambda(S)x \geq \sup S$.

(5) If Q is a subset of S, then $\Lambda(Q) \leq \Lambda(S)$.

(6) If T is a positive set and $S \cap T \neq \phi$, then $\Lambda(S \cup T) \leq \Lambda(\mathrm{S})\Lambda(\mathrm{T})$.

(7) If R is the set of reciprocals of elements of S, then $\Lambda(R) = \Lambda(S)$.

Proof. (1) follows from Lemma 2.8, which also implies (2) when $S \neq \emptyset$. (2) also holds when $S = \emptyset$ because $\Lambda(\emptyset) = \sup \emptyset = -\infty < \infty$ and $\inf \emptyset = \infty > 0$.

(3) We may assume $S \neq \emptyset$ because $\Lambda(\emptyset) = -\infty$. Since S is a finite, positive set, we have $\inf S = \min S > 0$ and $\sup S = \max S < \infty$, so $\Lambda(S) < \infty$ by (2).

(4) By Lemma 2.8, $\Lambda(S) = \sup S / \inf S$. Since $x \in \boldsymbol{R}^{+}$, the quotient $x/\Lambda(S)$ is defined. Therefore,

$$\frac{x}{\Lambda(S)} = \frac{x}{\sup S} \cdot \inf S \leq \inf S$$

and

$$\Lambda(S)x = \frac{x}{\inf S} \cdot \sup S \geq \sup S.$$

(5) $Q \subseteq S$ implies $\mathrm{Ratios}(Q) \subseteq \mathrm{Ratios}(S)$, and hence $\Lambda(Q) \leq \Lambda(S)$.

(6) If $\Lambda(S) = \infty$ or $\Lambda(T) = \infty$, then (5) implies $\Lambda(S \cup T) = \infty$; furthermore, $\Lambda(S) > 0$ and $\Lambda(T) > 0$ by (1), so $\Lambda(S)\Lambda(T) = \infty$ and

$$\Lambda(S \cup T) = \Lambda(S)\Lambda(T).$$

Now suppose $\Lambda(S)$ and $\Lambda(T)$ are finite. Let L_S and L_T be the greatest lower bounds of S and T, respectively, and let U_S and U_T be the least upper bounds of S and T, respectively. Since S and T are non-empty, we conclude that L_S and L_T are finite while U_S and U_T are positive. Therefore, part (2) implies $L_S, L_T, U_S, U_T \in \boldsymbol{R}^{+}$, and Lemma 2.8 implies that $\Lambda(\mathrm{S}) = U_S/L_S$ and $\Lambda(\mathrm{T}) = U_T/L_T$.

Define $a = \min(L_S, L_T)$, $A = \max(L_S, L_T)$, $b = \min(U_S, U_T)$, and $B = \max(U_S, U_T)$. Then $\inf(S \cup T) = a$, and $\sup(S \cup T) = B$.

There exists an element y common to S and T, so $A \le y \le b$. Therefore, $b/A \ge 1$, and Lemma 2.8 implies

$$\Lambda(S \cup T) = \frac{B}{a} \le \frac{b}{A}\frac{B}{a} = \frac{U_S}{L_S}\frac{U_T}{L_T} = \Lambda(S)\Lambda(T).$$

(The argument above for finite $\Lambda(S)$ and $\Lambda(T)$ (and $L_S, L_T, U_S, U_T \in \boldsymbol{R}^+$**)** is also valid when $\Lambda(S) = \infty$ or $\Lambda(T) = \infty$ according to our nonstandard conventions for arithmetic on $[0, \infty]$. Since the numerators are non-zero and the denominators are finite, the various fractions are defined. Furthermore, $b/A \ge 1$.)

(7) If x and y are elements of S, then $1/x$ and $1/y$ are elements of R, so that

$$\frac{x}{y} = \frac{1/y}{1/x} \in \text{Ratios}(R),$$

and hence $\text{Ratios}(S) \subseteq \text{Ratios}(R)$. Since S is the set of reciprocals of elements of R, we also have $\text{Ratios}(R) \subseteq \text{Ratios}(S)$. Therefore, $\text{Ratios}(R) = \text{Ratios}(S)$, which implies

$$\Lambda(R) = \Lambda(S).$$

□

Definition. If g is a positive function on a positive set D, define

$$\Psi_b(g) = \sup_{x \in D} \Lambda_g(D \cap [x, bx])$$

and

$$\text{Ratios}_b(g) = \{g(y)/g(x) : x, y \in D, x \le by, y \le bx\}$$

for all $b > 1$.

The simultaneous conditions $x \le by$ and $y \le bx$ are satisfied for $x, y \in D$ if and only if either $x \in [y, by]$ or $y \in [x, bx]$.

If f is a positive function on a positive, unbounded interval I, we obtain the slightly simpler representation

$$\Psi_b(f) = \sup_{x \in I} \Lambda_f([x, bx])$$

because $[x, bx] \subseteq I$ for all $x \in I$, i.e., $I \cap [x, bx] = [x, bx]$.

Trivialities about the empty function. $\Psi_b(\emptyset) = \sup \emptyset = -\infty$ and $\text{Ratios}_b(\emptyset) = \emptyset$ for all $b > 1$. In the trivial case of an empty function, Parts (7) and (8) of Lemma 2.10 will rely on our nonstandard convention that the products

$$(-\infty) \cdot (-\infty) = \infty$$

and

$$(-\infty) \cdot \infty = \infty \cdot (-\infty) = -\infty$$

are defined in $[-\infty, \infty]$.

The next lemma lists some of the most basic properties of $\Psi_b(g)$. The quotient in 2.10(2) uses our nonstandard convention that $x/0 = \infty$ for all non-zero $x \in (0, \infty]$. The quotient in 2.10(4) uses our nonstandard convention that $y/\infty = 0$ for all finite $y \in [0, \infty)$ (although we don't need the special case $0/\infty = 0$ here).

Lemma 2.10. If g is a positive function on a positive set D and $b > 1$, then

(1) $\Psi_b(g) = \sup \mathrm{Ratios}_b(g) = \sup_{S \in W} \Lambda_g(S)$ where $W = \{S \subseteq D : \Lambda(S) \leq b\}$.

(2) If D is non-empty, then

$$1 \leq \Psi_b(g) \leq \frac{\sup g(D)}{\inf g(D)}.$$

(3) $\Psi_b(g|_E) \leq \Psi_b(g)$ for each subset E of D.

(4) If $z \in S \subseteq D$ such that $\Lambda(S) \leq b$, then

$$\frac{g(z)}{\Psi_b(g)} \leq \inf g(S) \quad \text{and} \quad \Psi_b(g) g(z) \geq \sup g(S).$$

(5) $\Psi_b(1/g) = \Psi_b(g)$.

(6) $\Psi_a(g) \geq \Psi_b(g)$ for all $a \geq b$.

(7) If D is an interval, then $\Psi_{bc}(g) \leq \Psi_b(g)\Psi_c(g)$ for all $c > 1$.

(8) If D is an interval, then $\Psi_{b^n}(g) \leq \Psi_b(g)^n$ for all $n \in \boldsymbol{Z}^+$.

Proof. We conclude from

$$\bigcup_{S \in W} \mathrm{Ratios}\big(g(S)\big) = \mathrm{Ratios}_b(g) = \bigcup_{x \in D} \mathrm{Ratios}\big(g(D \cap [x, bx])\big)$$

that

$$\sup_{S \in W} \Lambda_g(S) = \sup_{S \in W}\Big(\sup \mathrm{Ratios}\big(g(S)\big)\Big) = \sup \mathrm{Ratios}_b(g)$$

and

$$\sup \mathrm{Ratios}_b(g) = \sup_{x \in D}\Big(\sup \mathrm{Ratios}\big(g(D \cap [x, bx])\big)\Big) = \sup_{x \in D} \Lambda_g(D \cap [x, bx]) = \Psi_b(g),$$

which proves (1). We now prove (2): Lemma 2.9(1) and the definition of $\Psi_b(g)$ imply $\Psi_b(g) \geq 1$. Let $L = \inf g(D)$ and $U = \sup g(D)$. Since D is non-empty and g is positive, we have $L \in [0, \infty)$ and $U \in (0, \infty]$, so U/L is defined. Furthermore, $g(t)/g(z) \leq U/L$ for all $t, z \in D$. Therefore,

$$\sup \mathrm{Ratios}_b(g) \leq \frac{U}{L},$$

which combines with (1) to show $\Psi_b(g) \leq U/L$, which proves (2).

Let $E \subseteq D$, so that $\mathrm{Ratios}_b(g|_E) \subseteq \mathrm{Ratios}_b(g)$, which combines with (1) to yield

$$\Psi_b(g|_E) = \sup \mathrm{Ratios}_b(g|_E) \leq \sup \mathrm{Ratios}_b(g) = \Psi_b(g),$$

which proves (3). Now let S and z be as in (4) and define $T = g(S)$, so $g(z) \in T$, which implies $T \neq \emptyset$. Part (1) and Lemma 2.9(1) imply $0 < \Lambda(T) \leq \Psi_b(g)$, which combines with $g(z) > 0$ and Lemma 2.9(4) to imply

$$\frac{g(z)}{\Psi_b(g)} \leq \frac{g(z)}{\Lambda(T)} \leq \inf T \leq \sup T \leq \Lambda(T) g(z) \leq \Psi_b(g) g(z),$$

which proves (4). Lemma 2.9(7) implies

$$\Lambda_{1/g}(D \cap [x, bx]) = \Lambda_g(D \cap [x, bx])$$

for all $x \in D$. (5) follows from (1) since $\mathrm{Ratios}_b(1/g) = \mathrm{Ratios}_b(g)$. Lemma 2.9(5) implies $\Lambda_g(D \cap [x, bx]) \leq \Lambda_g(D \cap [x, ax])$ for all $a \geq b$ and all $x \in D$, so

$$\Psi_b(g) = \sup_{x \in D} \Lambda_g(D \cap [x, bx]) \leq \sup_{x \in D} \Lambda_g(D \cap [x, ax]) = \Psi_a(g),$$

i.e., (6) holds. For the remainder of the proof, we assume D is an interval. If $D = \emptyset$, then

$$\Psi_{bc}(g) = -\infty < \infty = (-\infty)(-\infty) = \Psi_b(g)\Psi_c(g)$$

for all $c > 1$, and

$$\Psi_{b^n}(g) = -\infty \leq (-\infty)^n = \Psi_b(g)^n$$

for all $n \in \boldsymbol{Z}^+$. (Here $(-\infty)^n \in \{-\infty, \infty\}$.) Therefore, it suffices to prove assertions (7) and (8) under the assumption that $D \neq \emptyset$.

Let $c > 1$ and $x \in D$. Since D is an interval containing x, either $bx \in D$ or

$$D \cap [x, bcx] \subset [x, bx].$$

If $bx \in D$, then $g(bx)$ is defined and

$$g(bx) \in g(D \cap [x, bx]) \cap g(D \cap [bx, bcx]),$$

which implies

$$\Lambda_g(D \cap [x, bcx]) \leq \Lambda_g(D \cap [x, bx])\Lambda_g(D \cap [bx, bcx]) \leq \Psi_b(g)\Psi_c(g)$$

by Lemma 2.9(6). If instead $bx \notin D$, then Lemma 2.9(5) implies

$$\Lambda_g(D \cap [x, bcx]) \leq \Lambda_g(D \cap [x, bx]) \leq \Psi_b(g).$$

Part (2) implies $\Psi_c(g) \geq 1$, so

$$\Psi_{bc}(g) = \sup_{x \in D} \Lambda_g(D \cap [x, bcx]) \leq \Psi_b(g) \leq \Psi_b(g)\Psi_c(g)$$

as claimed in (7). Since

$$\Psi_{b^1}(g) = \Psi_b(g) = \Psi_b(g)^1,$$

assertion (8) is true for $n = 1$. Suppose $n \in Z^+$ such that $\Psi_{b^n}(g) \leq \Psi_b(g)^n$. It follows from (7) that

$$\Psi_{b^{n+1}}(g) \leq \Psi_{b^n}(g)\Psi_b(g) \leq \Psi_b(g)^n\Psi_b(g) = \Psi_b(g)^{n+1}.$$

Part (8) follows by induction. □

If $\Psi_b(g) = \infty$, Lemma 2.10(4) translates into the uninteresting inequalities

$$\inf g(D \cap [x, bx]) \geq 0 \quad \text{and} \quad \sup g(D \cap [x, bx]) \leq \infty$$

under our conventions for arithmetic on $[0, \infty]$. Similarly, assertions (7) and (8) become $\Psi_{bc}(g) \leq \infty$ and $\Psi_{b^n}(g) \leq \infty$, respectively.

Failure of $\boldsymbol{\Psi_{bc}(g) \leq \Psi_b(g)\Psi_c(g)}$ and $\boldsymbol{\Psi_{b^n}(g) \leq \Psi_b(g)^n}$ on disconnected domains. Define the function $g(x) = x$ on the set

$$D = \{e^n : n \in \mathbf{Z}^+\} = \{e, e^2, e^3, \ldots\}.$$

Since the domain D is not an interval, parts (7) and (8) of Lemma 2.10 are not applicable to the function g. Since $D \cap [x, 2x] = \{x\}$ for all $x \in D$, we obtain $\Psi_2(g) = 1$. Observe that $D \cap [e^n, 4e^n] = \{e^n, e^{n+1}\}$ for all $n \in \mathbf{Z}^+$, which implies $\Psi_4(g) = e$. Therefore,

$$\Psi_4(g) > \left(\Psi_2(g)\right)^2.$$

Lemma 2.11. If g is a positive function on a positive, unbounded interval and $b > 1$, then g is a b-polynomial-growth function if and only if $\Psi_b(g) < \infty$.

Proof. Let $I = domain(g)$. Since I is a positive, unbounded interval, we have $[x, bx] \subset I$ for all $x \in I$. Suppose $\Psi_b(g) < \infty$. Lemma 2.10(2) implies $\Psi_b(g)$ is positive, so we can define positive real numbers $c_1 = 1/\Psi_b(g)$ and $c_2 = \Psi_b(g)$. Lemma 2.10(4) implies

$$c_1 g(x) \leq g(u) \leq c_2 g(x)$$

for all $x \in I$ and all $u \in [x, bx]$. Therefore, g is a b-polynomial-growth function.

We now prove the converse. Suppose g is a b-polynomial-growth function, so there exist positive real numbers c_1 and c_2 such that

$$0 < c_1 g(x) \leq \inf g([x, bx]) \leq \sup g([x, bx]) \leq c_2 g(x) < \infty$$

for all $x \in I$. Lemma 2.8 implies

$$\Lambda_g([x, bx]) = \frac{\sup g([x, bx])}{\inf g([x, bx])} \leq \frac{c_2}{c_1}$$

for all such x. Therefore,

$$\Psi_b(g) = \sup_{x \in I} \left(\Lambda_g([x, bx])\right) \leq \frac{c_2}{c_1} < \infty.$$

□

Corollary 2.12. The function $x \mapsto x^\alpha$ on a positive set has polynomial growth for each real α.

Proof. Define $g: (0, \infty) \to \boldsymbol{R}^+$ by $g(x) = x^\alpha$, so $\Psi_2(g) = 2^{|\alpha|}$. Lemma 2.11 implies g is a 2-polynomial-growth function. By definition, $g|_D$ has polynomial growth for each positive set D. □

Corollary 2.13. If a real-valued function g on a positive set has a positive lower bound and finite upper bound, then g has polynomial growth.

Proof. Let L be a positive lower bound for g and let M be a finite upper bound for g. Define an extension h of g to $(0, \infty)$ by letting $g(z) = L$ for all $z \in (0, \infty) - D$ where D is the domain of g. Then $L \leq h(x) \leq M$ for all $x \in (0, \infty)$. Pick $b > 1$. By Lemma 2.10(1), we have

$$\Psi_b(h) = \sup\left(\text{Ratios}_b(h)\right) \leq \frac{M}{L} < \infty.$$

Lemma 2.11 implies h is a b-polynomial-growth function. By definition, g has polynomial growth. □

Corollary 2.14. Positive continuous functions on positive compact sets have polynomial growth.

Proof. Let g be a positive, continuous function on a positive compact set S. Since the empty function has polynomial growth, we may assume S is non-empty. Continuity of g and compactness of S implies g has a minimum and a maximum. The minimum and maximum are finite and positive because g is a positive real-valued function. Corollary 2.13 implies g has polynomial growth. □

Compactness condition of Corollary 2.14. Define the positive continuous function g on the positive half-open interval $(0, 1]$ by

$$g(x) = e^{1/x}.$$

If g has polynomial growth, then g has a b-polynomial-growth extension h for some real number $b > 1$. By definition of a b-polynomial-growth function, the domain of h is a positive, unbounded interval. (In this case, $domain(h) = (0, \infty)$.) Lemmas 2.10(3) and 2.11 imply

$$\Psi_b(g) \leq \Psi_b(h) < \infty.$$

However,

$$\Psi_b(g) \geq \lim_{x \to 0^+} \Lambda_g([x, bx]) = \lim_{x \to 0^+} \frac{e^{\frac{1}{x}}}{e^{\frac{1}{bx}}} = \lim_{x \to 0^+} e^{\frac{b-1}{bx}} = \lim_{t \to \infty} e^t = \infty,$$

which implies $\Psi_b(g) = \infty$ in contradiction to $\Psi_b(g) < \infty$. Thus g does not have polynomial growth. Corollary 2.14 is inapplicable to g because the domain of g is not compact.

Corollary 2.15. If g is a positive function, then g has polynomial growth if and only if $1/g$ has polynomial growth.

Proof. Suppose g has polynomial growth, so there exists a b-polynomial-growth extension h of g for some $b > 1$. The function h has polynomial growth because h is a b-polynomial-growth extension of itself. Since the empty function is its own reciprocal, we may assume g is non-empty. Therefore, positivity of g implies h is not identically zero. By Lemma 2.7, the function h is positive. Lemmas 2.10(5) and 2.11 imply

$$\Psi_b(1/h) = \Psi_b(h) < \infty.$$

The domain of $1/h$ is the same as the domain of h. In particular, $domain(1/h)$ is a positive, unbounded interval. Therefore, Lemma 2.11 implies $1/h$ is a b-polynomial-growth function. The function $1/g$ has polynomial growth because $1/h$ is an extension of $1/g$. The converse follows from the argument above because $1/g$ is positive and $g = 1/(1/g)$. □

Corollaries 2.12–2.15 can also be easily proved directly from the definitions of polynomial growth and b-polynomial-growth functions without appeal to Lemma 2.11.

The next proposition establishes independence of polynomial growth from the choice of $b > 1$.

Lemma 2.16. Let g be a real-valued function on a positive interval. Either all of conditions (1), (2), and (3) are true or all of them are false. If $domain(g)$ is unbounded, then either all of conditions (1) through (5) are true or all of them are false.

(1) g is a polynomial-growth function.
(2) Either g is identically zero, or g is positive and $\Psi_b(g) < \infty$ for *some* $b > 1$.
(3) Either g is identically zero, or g is positive and $\Psi_b(g) < \infty$ for *all* $b > 1$.

(4) g is a b-polynomial-growth function for *some* $b > 1$.
(5) g is a b-polynomial-growth function for *all* $b > 1$.

Proof. If g is identically zero, then (2) and (3) are satisfied a priori and (1) follows from Lemma 2.3, which also implies (4) and (5) when $domain(g)$ is unbounded. We conclude that the lemma is true when g is identically zero. Thus we may assume g is not identically zero. In particular, g is not the empty function, i.e., $domain(g) \neq \emptyset$.

If condition (1) is true, g can be extended to a b-polynomial-growth function h for some $b > 1$. By definition, the domain of h is a positive, unbounded interval. Lemma 2.2(4) implies h has polynomial growth. Since g is not identically zero, h is not identically zero. Lemma 2.7 implies g and h are positive. Lemmas 2.10(3) and 2.11 imply

$$\Psi_b(g) \leq \Psi_b(h) < \infty.$$

Therefore, (1) implies (2).

We now show that (2) implies (1). Suppose g is positive and $\Psi_b(g) < \infty$ for some $b > 1$. Lemma 2.10(2) implies $\Psi_b(g) \geq 1$. Let $I = domain(g)$ and $z = \sup I$. If $z = \infty$, then Lemmas 2.2(4) and 2.11 imply g has polynomial growth. Therefore, we may assume $z < \infty$. There exists a real number $t \in I \cap (z/b\,, z)$, so

$$I \cap [t, \infty) = I \cap [t, bt].$$

Let J be the minimum positive, unbounded interval containing I, so $J = I \cup [z, \infty)$. Define the positive function $f: J \to \boldsymbol{R}^+$ by $f|_I = g$ and $f(y) = g(t)$ for all $y \in J - I$. Let $x \in J$. If $x \notin I$, then $[x, bx] \subset J - I$, so $f([x, bx]) = \{g(t)\}$ and

$$\Lambda_f([x, bx]) = 1 \leq \Psi_b(g)$$

by Lemma 2.8. If $[x, bx] \subseteq I$, then

$$\Lambda_f([x, bx]) = \Lambda_g([x, bx]) \leq \Psi_b(g).$$

Assume $x \in I$ and $[x, bx] \nsubseteq I$, so $x \in I \cap [z/b\,, z]$ and $bx \geq z > t$. If $x \leq t$, then $t \in I \cap [x, bx]$, so

$$f([x,bx]-I)=\{g(t)\}\subseteq g(I\cap[x,bx]),$$

$$f([x,bx])=g(I\cap[x,bx])\cup f([x,bx]-I)=g(I\cap[x,bx]),$$

and

$$\Lambda_f([x,bx])=\Lambda_g(I\cap[x,bx])\leq\Psi_b(g).$$

Suppose instead that $x>t$, so

$$x\in I\cap[x,bx]\subset I\cap[t,\infty)=I\cap[t,bt].$$

Define the interval

$$K=\left[\frac{g(x)}{\Psi_b(g)},g(x)\Psi_b(g)\right].$$

Lemma 2.10(4) implies

$$f(I\cap[x,bx])=g(I\cap[x,bx])\subseteq K$$

and

$$f([x,bx]-I)=\{g(t)\}\subseteq g(I\cap[t,bt])\subseteq K.$$

Therefore, $f([x,bx])\subseteq K$. Lemmas 2.8 and 2.9(5) imply

$$\Lambda_f([x,bx])\leq\Lambda(K)=\left(\Psi_b(g)\right)^2.$$

Therefore,

$$\Psi_b(f)\leq\max\left(\Psi_b(g),\left(\Psi_b(g)\right)^2\right)=\left(\Psi_b(g)\right)^2<\infty.$$

Lemma 2.11 implies f is a b-polynomial-growth function. The function g satisfies the definition of a polynomial-growth function because f is an extension of g. Therefore, (2) implies (1). We conclude that (1) and (2) are equivalent, i.e., either both (1) and (2) are true or both are false.

We now show that (2) implies (3). Suppose again that $\Psi_b(g)<\infty$ for some $b>1$. Given $c>1$, there exists $n\in\boldsymbol{Z}^+$ such that $b^n\geq c$, so assertions (6) and (8) of Lemma 2.10 imply

$$\Psi_c(g)\leq\Psi_{b^n}(g)\leq\Psi_b(g)^n<\infty,$$

which confirms that (2) implies (3).

Condition (3) implies (2) because the interval $(1,\infty)$ is non-empty. Therefore, (2) and (3) are equivalent. Since conditions (1) and (2) are also equivalent, we conclude that conditions (1), (2), and (3) are equivalent.

Now suppose g has an unbounded domain. Lemmas 2.2(4), 2.7, and 2.11 imply (2) is equivalent to (4) and (3) is equivalent to (5). Therefore, either all or none of conditions (1) through (5) are true. □

We can now establish the relationship between Leighton's polynomial-growth condition and our definition of polynomial growth:

Corollary 2.17. A candidate g relative to a non-empty, finite subset S of $(0,1)$ has polynomial growth on $[\min S, \infty)$ if and only if g satisfies Leighton's polynomial-growth condition relative to S.

Proof. Let $c = \min S$ and $b = 1/c$, so $b > 1$. Lemma 2.16 implies g has polynomial growth on $[c, \infty)$ if and only if the restriction of g to $[c, \infty)$ is a *b-polynomial-growth* function, which Lemma 2.1 says is equivalent to g satisfying Leighton's polynomial-growth condition relative to S. □

Part of Lemma 2.16 is applicable to functions on arbitrary positive sets:

Corollary 2.18. If g is a positive polynomial-growth function, then $\Psi_b(g) < \infty$ for all $b > 1$.

Proof. By definition, g has a c-polynomial growth extension h for some $c > 1$. Since $domain(h)$ is a positive, unbounded interval, Lemmas 2.10(3) and 2.16 imply

$$\Psi_b(g) \leq \Psi_b(h) < \infty$$

for all $b > 1$. □

Example of $\boldsymbol{\Psi_b(g) < \infty}$ and $\boldsymbol{\Psi_c(g) = \infty}$ with $\boldsymbol{b > 1}$ and $\boldsymbol{c > 1}$. Define a function g on

$$D = \{3^n : n \in \boldsymbol{Z}^+\} = \{3,9,27, \dots\}$$

by $g(x) = e^x$. For all $x \in D$, we have $D \cap [x, 2x] = \{x\}$, so $\Psi_2(g) = 1$. However, $D \cap [x, 3x] = \{x, 3x\}$. Since D is unbounded and $g(3x)/g(x) = e^{2x}$ for all $x \in D$, we conclude that $\Psi_3(g) = \infty$. By Corollary 2.18, the function g does not have polynomial growth. Lemma 2.16 is inapplicable to g because D is not an interval.

Example of non-polynomial-growth function $\boldsymbol{g}$ with $\boldsymbol{\Psi_b(g) < \infty}$ for all $\boldsymbol{b > 1}$. Define a function g on

$$D = \{n! : n \in \boldsymbol{Z}^+\}$$

by $g(x) = e^x$. Let $b > 1$. The positive set $D \cap [x, bx]$ is finite for all $x \in D$, so Lemma 2.9(3) implies

$$\Lambda_g(D \cap [x, bx]) < \infty$$

for all such x. Furthermore, $D \cap [x, bx] \neq \{x\}$ for at most finitely many choices of $x \in D$. Since $\Lambda_g(\{x\}) = 1$, we conclude that $\Psi_b(g)$ is the supremum of a finite set of real numbers, which implies $\Psi_b(g) < \infty$. Later in this section (Corollary 2.35), we show that each polynomial-growth function f with $inf\ domain(f) > 0$ is bounded above by a corresponding function of the form cx^k for some positive real number c and some non-negative integer k. The function g clearly violates any such bound. We conclude that g does not have polynomial growth. Lemma 2.16 is inapplicable to g because D is not an interval.

The function domains in the preceding two examples and an earlier example, in which $\Psi_4(g) > \Psi_2(g)^2$, consist of positive, increasing infinite sequences. Discussion of polynomial-growth functions on such domains, which occur naturally in divide-and-conquer recurrences, can be found in Section 5.

Finite uniformity of b-polynomial-growth conditions. Let g be a polynomial-growth function on a positive, unbounded interval I. Lemma 2.16 implies g is a b-polynomial growth function for all $b > 1$. The definition of a b-polynomial-growth function requires the existence of $c_1 > 0$ and $c_2 > 0$ with certain properties. The choice of c_1 and c_2 depends on the choice of b. However, given a finite set of choices for b, a common choice of c_1 and c_2 exists. Let $b_1, \ldots, b_k$ be real numbers greater than 1 with maximum value b. There exist positive numbers c_1 and c_2 such that

$$c_1 g(x) \leq g(u) \leq c_2 g(x)$$

for all $x \in I$ and all $u \in [x, bx]$. Observe that $[x, b_i x] \subseteq [x, bx]$ for each i, so

$$c_1 g(x) \leq g(w) \leq c_2 g(x)$$

for all $w \in [x, b_i x]$.

Non-negativity. For completeness, we consider what happens if the assumption of non-negativity is dropped. For each $b > 1$, define a *weak b-polynomial-*growth function to be a real-valued function that satisfies all the requirements of a *b-polynomial-*growth function except it need not be non-negative. Also define a function to have *weak polynomial growth* if it can be extended to a weak b-polynomial-growth function for some $b > 1$. With obvious modifications, the arguments of Lemmas 2.4–2.7 show that a weak polynomial-growth function must be positive, negative, or identically zero. For each $b > 1$, a negative function on a positive, unbounded interval is a weak b-polynomial-growth function if and only if its negative is a b-polynomial-growth function, i.e., a polynomial-growth function (Lemma 2.16). It follows that a negative function has weak polynomial growth if and only if its negative has polynomial growth.

We can also define a weak version of Leighton's polynomial-growth condition that allows a *weak candidate* to *weakly satisfy* Leighton's polynomial-growth condition: Insert "weak" before "candidate" and "weakly" before "satisfies". The definition of a *weak candidate* is obtained from the definition of a candidate by deleting the requirement for non-negativity. A variant of Lemma 2.1 can be obtained by inserting "weakly" before "satisfies" and "weak" before "candidate" and "b-polynomial-growth". The resulting proposition implies a variant of Corollary 2.17 that inserts "weakly" before "satisfies" and "weak" before "candidate" and "polynomial growth".

The next lemma is primarily of interest when the domain of a polynomial-growth function is (c, ∞) for some $c > 0$.

Lemma 2.19. If g is a polynomial-growth function, and $c \in (0, \infty)$ is a limit point of $domain(g)$, then

$$\limsup_{x \to c} g(x) < \infty.$$

If g is positive, then

$$\liminf_{x \to c} g(x) > 0.$$

(The limits are taken as elements of $domain(g)$ approach c.)

Proof. By Lemma 2.2(5), g has a polynomial-growth extension h to some positive, unbounded interval I, which also has c as a limit point. We conclude that I contains (c, ∞).

If g is identically zero, the limit superior in question is zero, and hence finite; the hypothesis for the second inequality is not satisfied. Thus we may assume g is not identically zero. Therefore, h is not identically zero. Lemmas 2.7, 2.10(2), and 2.16 imply h is positive and $0 < \Psi_2(h) < \infty$.

Since $c \in (0, \infty)$, the open interval $W = (3c/4, 2c)$ is non-empty and contains c as an interior point. Furthermore, $3c/2$ and $2c$ are elements of (c, ∞) and are therefore in I, the domain of h. Define the positive real numbers

$$L_1 = \frac{h(3c/2)}{\Psi_2(h)},$$

$$U_1 = \Psi_2(h) h(3c/2),$$

$$L_2 = \frac{h(2c)}{\Psi_2(h)},$$

and

$$U_2 = \Psi_2(h) h(2c).$$

Suppose $x \in I \cap W$ such that $x \neq c$. Observe that $2x > 3c/2$ and $x < 2c$. If $x < c$, then $3c/2 \in I \cap [x, 2x]$, so Lemma 2.10(4) implies

$$L_1 \leq h(x) \leq U_1.$$

If $x > c$, then $2c \in I \cap [x, 2x]$, so Lemma 2.10(4) implies

$$L_2 \leq h(x) \leq U_2.$$

Therefore,

$$\limsup_{x \to c} g(x) \leq \limsup_{x \to c} h(x) \leq \max(U_1, U_2) < \infty$$

and

$$\liminf_{x \to c} g(x) \geq \liminf_{x \to c} h(x) \geq \min(L_1, L_2) > 0.$$

□

Counterexamples at zero and infinity. The requirement of Lemma 2.19 that $c \in (0, \infty)$ is essential. Corollary 2.12 implies the positive functions $x \mapsto x$ and $x \mapsto 1/x$ on $(0, \infty)$ have polynomial growth. Their domains have 0 and ∞ as limit points in $[0, \infty]$. However,

$$\limsup_{x \to \infty} x = \lim_{x \to \infty} x = \infty,$$

$$\liminf_{x \to 0^+} x = \lim_{x \to 0} x = 0,$$

$$\limsup_{x \to 0^+} 1/x = \lim_{x \to 0^+} 1/x = \infty,$$

and

$$\liminf_{x \to \infty} 1/x = \lim_{x \to \infty} 1/x = 0.$$

Positive polynomial function without polynomial growth. Define the polynomial function $g(x) = x - 1$ on $(1, \infty)$. The function g is positive and

$$\lim_{x \to 1^+} g(x) = 0.$$

Lemma 2.19 implies g does not have polynomial growth. Since

$$\lim_{x \to 1^+} 1/g(x) = \infty,$$

Lemma 2.19 implies $1/g$ does not have polynomial growth either. (Non-polynomial-growth of $1/g$ also follows from Corollary 2.15.) Failure of g and $1/g$ to be polynomial-growth functions is illustrated by

$$\Psi_2(1/g) = \Psi_2(g) \geq \limsup_{x \to 1^+} \frac{g(2x)}{g(x)} = \lim_{x \to 1^+} \frac{2x - 1}{x - 1} = \infty.$$

(See Lemmas 2.10(5) and 2.16.)

Sensitivity of polynomial growth to domain. As in the preceding example, define the function $g(x) = x - 1$ on $(1, \infty)$. Although g and $1/g$ do not have polynomial growth on $(1, \infty)$, they have polynomial growth on each positive, unbounded interval I properly contained in $(1, \infty)$: Let $c = \inf I$, so $c > 1$. The function g is increasing (and positive), so

$$\Psi_2(1/(g|_I)) = \Psi_2(g|_I) = \sup_{x \in I} \frac{g(2x)}{g(x)} = \sup_{x \in I} \frac{2x - 1}{x - 1}.$$

The function $(2x - 1)/(x - 1)$ is continuous and decreases on $(1, \infty)$, so

$$\sup_{x \in I} \frac{2x-1}{x-1} = \frac{2c-1}{c-1} < \infty$$

as required by Lemma 2.16.

Lemma 2.20. If g is a positive function on a set S of real numbers with $\inf S > 0$ and $\sup S < \infty$, then g has polynomial growth if and only if $g = \Theta(1)$.

Proof. If $g = \Theta(1)$, then g has polynomial growth by Corollary 2.13. We now prove the converse. Suppose g has polynomial growth.

The Lemma is vacuously true for the empty function, so we may assume S is non-empty. Let $c = \inf S$ and $d = \sup S$, so $0 < c \leq d < \infty$ and $S \subseteq [c, d]$.

We claim g is $\Theta(1)$ on $S \cap [a, d]$ for all $a \in S$: If $a = d$, then $S \cap [a, d] = \{a\}$, so $g(a)$ is simultaneously a positive lower bound and finite upper bound for the restriction of g to $S \cap [a, d]$, which implies g is $\Theta(1)$ on $S \cap [a, d]$. Therefore, we may assume $a < d$. Let $b = d/a$ so $b > 1$. Lemma 2.10(2) and Corollary 2.18 imply $0 < \Psi_b(g) < \infty$. Lemma 2.10(4) implies $g(a)/\Psi_b(g)$ and $\Psi_b(g)g(a)$ are a positive lower bound and finite upper bound, respectively, for the restriction of g to $S \cap [a, d]$. In particular, g is $\Theta(1)$ on $S \cap [a, d]$ as claimed.

If $c \in S$, then g is $\Theta(1)$ on $S \cap [c, d]$, i.e., g is $\Theta(1)$ on S. Therefore, we may assume $c \notin S$. We conclude from $S \subseteq [c, d]$ and $S \neq \phi$ that $c \neq d$, so $c < d$ and $S \subseteq (c, d]$. Furthermore, c is a limit point for S. Define

$$L = \frac{1}{2} \cdot \liminf_{x \to c} g(x)$$

and

$$U = 2 \cdot \limsup_{x \to c} g(x)$$

where the limits are taken as elements of $domain(g)$ approach c. Lemma 2.19 implies $L > 0$ and $U < \infty$. There exist $v, w \in S \cap (c, d)$ such that $g(y) \geq L$ and $g(z) \leq U$ for all $y \in S \cap (c, v)$ and all $z \in S \cap (c, w)$. Let $t = \min(v, w)$ so $t \in S \cap (c, d)$ and

$$L \leq g(x) \leq U$$

for all $x \in S \cap (c, t)$. In particular, g is $\Theta(1)$ on $S \cap (c, t)$. As we previously established, g is also is $\Theta(1)$ on $S \cap [t, d]$. Therefore g is $\Theta(1)$ on

$$\big(S \cap (c, t)\big) \cup (S \cap [t, d]) = S \cap (c, d] = S.$$

□

Requirement that inf $S > 0$. The condition $\inf S > 0$ of Lemma 2.20 cannot be replaced with positivity of S, as illustrated by the polynomial growth of functions x and

$1/x$ on $(0,1)$ implied by Corollary 2.12. They lack a positive lower bound and finite upper bound, respectively.

For future reference, we list three obvious corollaries to Lemma 2.20:

Corollary 2.21. If g is a positive polynomial-growth function, then $g|_S = \Theta(1)$ for all $S \subseteq domain(g)$ with $\inf S > 0$ and $\sup S < \infty$.

Proof. The function $g|_S$ inherits positivity from g. Lemma 2.2(2) implies $g|_S$ has polynomial growth, so $g|_S = \Theta(1)$ by Lemma 2.20. □

Corollary 2.22. A positive polynomial-growth function is locally $\Theta(1)$ if its domain has a positive lower bound.

Proof. Let g be a positive polynomial-growth function. If S is a bounded subset of $domain(g)$, then $\inf S \geq inf\ domain(g) > 0$ and $\sup S < \infty$, so $g|_S = \Theta(1)$ by Corollary 2.21. Therefore, g is locally $\Theta(1)$. □

Corollary 2.23. A polynomial-growth function g is bounded on each subset S of its domain that satisfies $\inf S > 0$ and $\sup S < \infty$.

Proof. By Lemma 2.7, g is either positive or identically zero. If g is positive, then Corollary 2.21 implies $g|_S = \Theta(1)$; in particular, $g|_S$ is bounded. If g is identically zero, than $g|_S$ is bounded above and below by zero. □

Lemma 2.24. Let $D = A \cup B$ be a positive set, where one of A, B is either a lower or upper subset of D. A positive function g on D has polynomial growth (on D) if and only if g has polynomial growth on A and B.

Proof. If g is a polynomial-growth function, then Lemma 2.2(2) implies g has polynomial growth on A and B. We now prove the converse. We suppose g has polynomial growth on both A and B and will show that g is a polynomial-growth function (i.e., has polynomial growth on all of D).

If one of A, B is a lower subset of D, we may assume without loss of generality that A is a lower subset. If neither A nor B is a lower subset, then one of A, B is an upper subset, which we may assume is B without loss of generality.

If A is a lower subset, define $L = A$ and $U = D - A$. Otherwise B is an upper subset, and we define $L = D - B$ and $U = B$. The set D is the disjoint union of L and U. Furthermore, L is a lower subset of D, and U is an upper subset of D. Observe that $L \subseteq A$ and $U \subseteq B$. By Lemma 2.2(2), polynomial growth of g on A and B implies polynomial growth of g on L and U, respectively.

If $L = \emptyset$, then $D = U$, which combines with polynomial growth of $g|_U$ to imply g is a polynomial-growth function. Similarly, if $U = \emptyset$, then $D = L$, which combines with polynomial growth of $g|_L$ to imply g is a polynomial-growth function. Thus we may assume L and U are non-empty, so D is also non-empty.

Let D^* and U^* be the minimum, positive, unbounded intervals containing D and U, respectively. The containment $U \subseteq D$ implies $U^* \subseteq D^*$. Since L is a non-empty lower subset of D, we conclude that D^* is also the minimum positive, unbounded interval containing L. Define the interval $L^* = D^* - U^*$, so L^* is a lower subset of D^* and U^* is an upper subset of D^*. For all $a \in L$, we have $a < b$ for all $b \in U$, so $U \subseteq (a, \infty)$ and hence $U^* \subseteq (a, \infty)$, which implies $\inf U^* \geq a > 0$ and $a \notin U^*$, so $a \in L^*$. Therefore, $L \subseteq L^* \subset D^*$ (and $L^* \neq \emptyset$, $U^* \subset D^*$).

By Lemma 2.2(6), there exist polynomial-growth extensions f of $g|_L$ and h of $g|_U$ to D^* and U^*, respectively. Positivity of g and non-emptiness of L and U imply neither f nor h is identically zero. Lemma 2.7 implies f and h are positive. By Lemma 2.2(2), $f|_{L^*}$ has polynomial growth. Since D^* is the disjoint union of L^* and U^*, we may define a positive function p on D^* by $p|_{L^*} = f|_{L^*}$ and $p|_{U^*} = h$, so that p has polynomial growth on L^* and U^*. Corollary 2.18 implies $\Psi_2(p|_{L^*}) < \infty$ and $\Psi_2(p|_{U^*}) < \infty$. Furthermore,

$$p|_L = (p|_{L^*})|_L = (f|_{L^*})|_L = f|_L = g|_L$$

and

$$p|_U = (p|_{U^*})|_U = h|_U = g|_U$$

because $L \subseteq L^* \subset D^*$ and $U \subseteq U^* \subset D^*$. Therefore,

$$p|_D = p|_{L \cup U} = g|_{L \cup U} = g|_D = g.$$

Let $c = \inf U^*$ and $W = D^* \cap [c/2, 2c]$, so $\inf W \geq c/2 > 0$ and $\sup W \leq 2c < \infty$. Define $S = L^* \cap W$ and $T = U^* \cap W$, so that $W = S \cup T$. The sets S and T have positive lower bounds and finite upper bounds. Since $S \subseteq L^* \subset D^*$ and $T \subset U^*$, Lemma 2.2(2) implies f has polynomial growth on S, and h has polynomial growth on T. Observe that

$$p|_S = (p|_{L^*})|_S = (f|_{L^*})|_S = f|_S$$

and

$$p|_T = (p|_{U^*})|_T = h|_T,$$

so p has polynomial growth on S and T. Lemma 2.20 implies $p|_S$ and $p|_T$ are $\Theta(1)$. Since $W = S \cup T$, the function $p|_W$ is also $\Theta(1)$, i.e., $p|_W$ has a positive lower bound and finite upper bound. (Recall our definition of $\Theta(1)$ on a set with a finite upper bound). Corollary 2.13 implies p has polynomial growth on W, and Corollary 2.18 implies $\Psi_2(p|_W)$ is finite.

Let $x \in D^*$, so $[x, 2x] \subseteq D^*$. If $[x, 2x] \subseteq L^*$, then

$$\Lambda_p([x, 2x]) \leq \Psi_2(p|_{L^*}).$$

If $[x, 2x] \subseteq U^*$, then

$$\Lambda_p([x, 2x]) \le \Psi_2(p|_{U^*}).$$

If $[x, 2x]$ is not contained in either of the intervals L^* or U^*, we conclude from $D^* = L^* \cup U^*$ and $\sup L^* \le \inf U^*$ that $x \in L^*$ and $2x \in U^*$. Thus $x \le c \le 2x$, so $c/2 \le x$ and $2x \le 2c$. We conclude that $[x, 2x] \subset [c/2, 2c]$, so $[x, 2x] \subseteq W$ and

$$\Lambda_p([x, 2x]) \le \Psi_2(p|_W).$$

Therefore,

$$\Psi_2(p) \le \max\big(\Psi_2(p|_{L^*}), \Psi_2(p|_{B^*}), \Psi_2(p|_W)\big) < \infty.$$

Lemma 2.16 implies p has polynomial growth, so $p|_D$ has polynomial growth by Lemma 2.2(2). The proposition follows from $p|_D = g$. □

Counterexample when g is not a positive function. Define $D = \{1,2\}$, $A = \{1\}$, and $B = \{2\}$, so A is a lower subset of D and B is an upper subset of D. Define $g: D \to \boldsymbol{R}$ by $g(1) = 0$ and $g(2) = 1$. Lemma 2.3 implies $g|_A$ and $g|_B$ have polynomial growth, and Lemma 2.7 implies g is not a polynomial-growth function. Lemma 2.24 is inapplicable to g because g is not a positive function.

Counterexample when neither A nor B is a lower or upper subset of the domain. Let A be the set of odd positive integers and let B be the set of even positive integers. Define a positive function g on $\boldsymbol{Z}^+$ by $g(a) = 1$ and $g(b) = b$ for all $a \in A$ and each $b \in B$. Lemmas 2.3 and Corollary 2.12 imply g has polynomial growth on A and B. Observe that $\Psi_2(g) = \infty$. By Corollary 2.18, g is not a polynomial-growth function. Lemma 2.24 is inapplicable since neither A nor B is a lower or upper subset of $\boldsymbol{Z}^+$.

We now identify some simple consequences of Lemma 2.24:

Corollary 2.25. Let $D = A \cup B$ be a positive set, where one of A, B is either a lower or upper subset of D. If $\inf A > 0$ and $\sup A < \infty$, then a positive function g on D has polynomial growth (on D) if and only if $g|_A = \Theta(1)$ and g has polynomial growth on B.

Proof. By Lemma 2.24, g has polynomial growth (on D) if and only if g has polynomial growth on A and B. Lemma 2.20 implies implies g has polynomial growth on A if and only if only if $g|_A = \Theta(1)$. □

Corollary 2.26. Let d be an element of a positive set D. A positive function g on D has polynomial growth (on D) if and only if g has polynomial growth on $D - \{d\}$.

Proof. If g is a polynomial-growth function, then g has polynomial growth on $D - \{d\}$ by Lemma 2.2(2). We now prove the converse. Assume g has polynomial growth on $D - \{d\}$.

Lemma 2.3 implies g has polynomial growth on $\{d\}$. Let $L = \{x \in D : x < d\}$ and $U = \{x \in D : x > d\}$. Lemma 2.2(2) implies g has polynomial growth on L and U since L and U are subsets of $D - \{d\}$. The singleton $\{d\}$ is a lower subset of $\{d\} \cup U$, so Lemma 2.24 implies g has polynomial growth on $\{d\} \cup U$, which is an upper subset of D. Since $D = L \cup (\{d\} \cup U)$, Lemma 2.24 implies g is a polynomial-growth function. □

Polynomial growth of a positive function on $[c, \infty)$ vs. (c, ∞) with $c > 0$. By Corollary 2.26, a positive function g on $[c, \infty)$ has polynomial growth if and only g has polynomial growth on (c, ∞).

Corollary 2.27. Suppose $c > 0$, and g is a real-valued function on $[c, \infty)$ that is continuous at c. The function g has polynomial growth on $[c, \infty)$ if and only if g has polynomial growth on (c, ∞).

Proof. Let h be the restriction of g to (c, ∞). If g is a polynomial-growth function then h has polynomial growth by Lemma 2.2(2). We now prove the converse. Suppose h has polynomial growth. Lemma 2.7 implies h is either positive or identically zero. If h is identically zero, then continuity of g at c implies $g(c) = 0$, so g is identically zero and has polynomial growth by Lemma 2.3. If h is positive, then Lemma 2.19 and continuity of g at c imply $g(c) > 0$, so g is positive. Polynomial growth of g follows from Corollary 2.26. □

Corollary 2.28. Let $D = A \cup B$ be a positive set, where A is finite. A positive function g on D has polynomial growth (on D) if and only if g has polynomial growth on B.

Proof. If g is a polynomial-growth function, then g has polynomial growth on B by Lemma 2.2(2). We now prove the converse. Suppose g has polynomial growth on B. The set

$$S = \{W \subseteq A : \ g \text{ has polynomial growth on } W \cup B\}$$

is non-empty because $\emptyset \in S$. Furthermore, S is finite because it is a subset of the finite power set 2^A. Suppose $V \in S - \{A\}$, so $V \subset A$, i.e., there exists $a \in A - V$. The function g has polynomial growth on $V \cup B$ because $V \in S$. Corollary 2.26 implies g has polynomial growth on $V \cup B \cup \{a\}$. Then $V \cup \{a\} \in S$ since $V \cup \{a\} \subseteq A$.

The set S is partially ordered by set containment ($\subseteq$). Since S is finite and non-empty, S must contain a maximal element A^* relative to set containment. We demonstrated that all elements of S other than A are non-maximal. Therefore, $A^* = A$, which implies $A \in S$, i.e., g has polynomial growth on the set $D = A \cup B$. □

Corollary 2.29. Positive functions on finite, positive sets have polynomial growth.

Proof. Let g be a positive function on a finite, positive set D. Observe that $D = D \cup \emptyset$. The restriction of g to the empty set has polynomial growth by Lemma 2.3. Corollary 2.28 implies g is a polynomial-growth function. (The proposition also follows directly from Corollary 2.13.) □

Corollary 2.30. If g is a polynomial-growth function and S is a positive set containing the domain of g, then g can be extended to a polynomial-growth function on S.

Proof. By Lemma 2.3, the identically zero function z on S has polynomial growth. If g is identically zero, then z is an extension of g. Therefore, we may assume g is not identically zero. By Lemma 2.2(5), g can be extended to a polynomial-growth function h on some positive, unbounded interval I containing the domain, D, of g. The function h is not identically zero because g is not identically zero. Lemma 2.7 implies h is positive.

Define a function f on $(0,\infty)$ by $f|_I = h$ and $f(x) = 1$ for all $x \in (0,\infty) - I$, so f has polynomial growth on I. Lemma 2.3 implies f has polynomial growth on $(0,\infty) - I$. The function f is positive, and I is an upper subset of $(0,\infty)$. Furthermore, the domain of f is the union of I and $(0,\infty) - I$. Lemma 2.24 implies f has polynomial growth. By hypothesis, S is a positive set, i.e., $S \subseteq (0,\infty) = domain(f)$. The restriction of f to S has polynomial growth by Lemma 2.2(2) and is an extension of g to S because

$$(f|_S)|_D = f|_D = (f|_I)|_D = h|_D = g.$$

□

The simple observation below further illustrates the connection between Leighton's polynomial-growth condition and our definition of polynomial growth:

Corollary 2.31. If g is a polynomial-growth function and S is a non-empty, finite subset of $(0,1)$, then g can be extended to a polynomial-growth function that satisfies Leighton's polynomial-growth condition relative to S.

Proof. Corollary 2.30 implies g can be extended to a polynomial-growth function h on the interval $(0,\infty)$, which contains $[\min S, \infty)$. Lemma 2.2(1) implies h is non-negative, so h is a candidate for Leighton's polynomial-growth condition relative to S. By Lemma 2.2(2), h has polynomial growth on $[\min S, \infty)$. Corollary 2.17 implies g satisfies Leighton's polynomial-growth condition relative to S. □

Polynomial growth is preserved by Θ-equivalence of sufficiently nice positive functions:

Lemma 2.32. Suppose $g = \Theta(h)$, where g is a real-valued function on a positive, unbounded set, and h is a positive polynomial-growth function. If g is locally $\Theta(1)$, then g has polynomial growth. The converse is true if $domain(g)$ has a positive lower bound.

Proof. Since $g = \Theta(h)$, there exists a positive, unbounded interval I and $\alpha, \beta \in \boldsymbol{R}^+$ such that

$$\emptyset \neq D \subseteq E$$

and

$$\alpha h(x) \leq g(x) \leq \beta h(x)$$

for all $x \in D$, where $D = domain(g) \cap I$ and $E = domain(h) \cap I$. Positivity of h implies positivity of $g|_D$. Let $a = \min(\alpha, 1)$ and $b = \max(\beta, 1)$, so that $0 < a \leq 1 \leq b$ and

$$ah(x) \leq \alpha h(x) \leq g(x) \leq \beta h(x) \leq bh(x)$$

for all $x \in D$. By Lemma 2.2(2) and Corollary 2.30, the restriction of h to E has polynomial growth and can be extended to a polynomial-growth function h^* on I. Since h is positive and $E \neq \emptyset$, we conclude that h^* is not identically zero. Lemma 2.7 implies h^* is positive. Since $D \subseteq E$, the function h^* agrees with h on D. Let g^* be the function on I that agrees with g on D and agrees with h^* on $I - D$. Positivity of $g|_D$ and h^* implies g^* is positive. Furthermore,

$$ah^*(v) = ah(v) \leq g(v) = g^*(v) = g(v) \leq bh(v) = bh^*(v)$$

for all $v \in D$. Since $a \leq 1 \leq b$, we have

$$ah^*(w) = ag^*(w) \leq g^*(w) \leq bg^*(w) = bh^*(w)$$

for all $w \in I - D$. Therefore,

$$ah^*(t) \leq g^*(t) \leq bh^*(t)$$

for all $t \in I$. Lemmas 2.10(1) and 2.16 imply

$$\Psi_2(g^*) = \sup \mathrm{Ratios}_2(g^*) \leq \frac{b}{a} \cdot (\sup \mathrm{Ratios}_2(h^*)) = \frac{b}{a} \cdot \Psi_2(h^*) < \infty$$

and g^* has polynomial growth. Since $g|_D = g^*|_D$, Lemma 2.2(2) implies $g|_D$ has polynomial growth.

Suppose g is locally $\Theta(1)$, so g is a positive function. Define $S = domain(g) - D$, so S is bounded below by zero. Furthermore, S is bounded above because D is a non-empty upper subset of $domain(g)$. Therefore, g is $\Theta(1)$ on S, so Corollary 2.13 implies g has polynomial growth on S. Since g has polynomial growth on S and D, Lemma 2.24 implies g is a polynomial-growth function.

Conditional converse: Suppose g has polynomial growth and $\inf domain(g) > 0$. Since $g|_D$ is positive and $D \neq \emptyset$, the function g is not identically zero. Lemma 2.7 implies g is positive. Then g is locally $\Theta(1)$ by Corollary 2.22. □

Example. Define the positive function $g(x) = x + \sin x$ on $(2, \infty)$, so

$$x - 1 \leq g(x) \leq x + 1.$$

If $S \subseteq (2, \infty)$ is bounded, then

$$1 \leq \inf S - 1 \leq g(t) \leq \sup S + 1$$

for all $t \in S$, i.e., $g|_S = \Theta(1)$. Thus g is locally $\Theta(1)$. Let h be the identity function, $h(x) = x$, on $(2, \infty)$. Corollary 2.12 implies h has polynomial growth. Since $g = \Theta(h)$, Lemma 2.32 implies g has polynomial growth. Of course, polynomial growth of g also follows from

$$\Psi_2(g) \leq \sup_{x>2} \frac{2x+1}{x-1} = \lim_{x \to 2} \frac{2x+1}{x-1} = 5 < \infty.$$

Asymptotic polynomial growth does not imply polynomial growth. Let h be a positive polynomial-growth function on $(1, \infty)$. Define a function g on $(1, \infty)$ by $g(x) = h(x)$ for $x \geq 2$, and $g(x) = 1/(x-1)$ for $1 < x < 2$. The function g is unbounded above on the bounded interval (1,2), so g is not locally $\Theta(1)$. Although $g = \Theta(h)$, Lemma 2.32 implies g is not a polynomial-growth function. However, Lemma 2.32 implies g has polynomial growth on each subset of $(1, \infty)$ that does not have 1 in its closure.

$\Theta(0)$. Lemmas 2.3 and 2.7 imply a polynomial growth function g on an unbounded set satisfies $g = \Theta(0)$ if and only if g is identically zero.

Lemma 2.32 has a particularly simple interpretation for continuous functions on positive, unbounded, closed intervals:

Corollary 2.33. If g is a continuous real-valued function on $[c, \infty)$ for some $c > 0$, and $g = \Theta(h)$ for some positive polynomial-growth function h, then g has polynomial growth if and only if g is positive.

Proof. It follows from $g = \Theta(h)$ and positivity of h that g is not identically zero. If g has polynomial growth, then g is positive by Lemma 2.7.

We now prove the converse. Suppose g is positive. If S is a bounded subset of the closed set $[c, \infty)$, then the closure $\bar{S}$ of S is also a bounded subset of $[c, \infty)$. Continuity of g implies the restriction of g to $\bar{S}$ has a minimum L and maximum U. We have $L > 0$ and $U < \infty$ since g is a positive real-valued function. Since $S \subseteq \bar{S}$, The quantities L and U are a lower and upper bound, respectively, for the restriction of g to S. Therefore, g is $\Theta(1)$ on S. We conclude that g is locally $\Theta(1)$. Lemma 2.32 implies g has polynomial growth. □

Examples of domain requirements. The positive, continuous function $f(x) = x - 1$ on $(1, \infty)$ satisfies $f(x) = \Theta(x)$. As explained after Lemma 2.19, f is not a polynomial-growth function. Corollary 2.33 is inapplicable to f because the domain of f is $(1, \infty)$. However, Corollary 2.33 implies the restriction of f to $[c, \infty)$ has polynomial growth for all $c > 1$.

The positive, continuous function g on $(0, \infty)$ defined by

$$g(x) = \begin{cases} e^{1/x}, & \text{for } x \leq 1 \\ x + e - 1, & \text{for } x > 1 \end{cases}$$

satisfies $g(x) = \Theta(x)$. Corollary 2.33 is inapplicable to g because the domain is $(0, \infty)$. We have

$$\Psi_2(g) \geq \lim_{x \to 0^+} \frac{e^{1/x}}{e^{1/(2x)}} = \lim_{x \to 0^+} \sqrt{e^{1/x}} = \infty,$$

so $\Psi_2(g) = \infty$. Lemma 2.16 implies g does not have polynomial growth. However, Corollary 2.33 implies g has polynomial growth on $[c, \infty)$ for all $c > 0$.

The polynomial growth functions $h(x) = x$ and $k(x) = 1/x$ on $(0, \infty)$ trivially satisfy $h(x) = \Theta(x)$ and $k(x) = \Theta(1/x)$. Since $(0, \infty)$ is the domain of h and k, we cannot conclude from Lemma 2.32 that h and k are locally $\Theta(1)$. Indeed, they are not $\Theta(1)$ on the bounded set $(0,1)$.

We now show that polynomial growth has something to do with polynomials:

Lemma 2.34. If f is a positive polynomial-growth function and $domain(f)$ has a positive lower bound, then there exists a non-negative integer n and positive real numbers α and β such that

$$\alpha x^{-n} \leq f(x) \leq \beta x^n.$$

for all $x \in domain(f)$.

Proof. The assertion is vacuously satisfied by the empty function with $\alpha = \beta = n = 1$, so we may assume f is non-empty. Let $c = inf\ domain(f)$, so $0 < c < \infty$ and $domain(f) \subseteq [c, \infty)$. By Corollary 2.30, the function f can be extended to a polynomial-growth function g on $[c, \infty)$. Let $b > 1$ be a real number. Lemmas 2.10(2) and 2.16 imply

$$1 \leq \Psi_b(g) < \infty,$$

so $\log_b(\Psi_b(g))$ is a non-negative real number. Define the non-negative integer

$$n = \lceil \log_b(\Psi_b(g)) \rceil,$$

so

$$\Psi_b(g) \leq b^n.$$

Let

$$\alpha = \frac{c^n}{\Psi_b(g)}\, g(c)$$

and

$$\beta = \frac{\Psi_b(g)}{c^n}\, g(c).$$

It follows from $\Psi_b(g) \geq 1$ that

$$\alpha c^{-n} \leq g(c) \leq \beta c^n.$$

Now let $x > c$, and define the positive integer

$$L = \lceil \log_b(x/c) \rceil,$$

so

$$b^L \geq \frac{x}{c} > 1.$$

Therefore, $\Psi_{b^L}(g)$ and $\Psi_{x/c}(g)$ are defined. Observe that

$$L < 1 + \log_b(x/c),$$

so parts (2), (6) and (8) of Lemmas 2.10 imply

$$\Psi_{x/c}(g) \leq \Psi_{b^L}(g) \leq \left(\Psi_b(g)\right)^L \leq \Psi_b(g)\left(\Psi_b(g)\right)^{\log_b(x/c)} \leq \Psi_b(g) \cdot b^{n \cdot \log_b(x/c)},$$

which implies

$$\Psi_{x/c}(g) \leq \Psi_b(g)\left(\frac{x}{c}\right)^n.$$

Lemma 2.10(4) combines with the inequality above to imply

$$\alpha x^{-n} = \frac{g(c)}{\Psi_b(g)}\left(\frac{c}{x}\right)^n \leq \frac{g(c)}{\Psi_{x/c}(g)} \leq \inf g([c,x]) \leq g(x)$$

and

$$\beta x^n = \Psi_b(g)\left(\frac{x}{c}\right)^n g(c) \geq \Psi_{x/c}(g)g(c) \geq \sup g([c,x]) \geq g(x).$$

The lemma follows because g is an extension of f. □

For convenience, we identify a trivial consequence of Lemma 2.34 that includes indentically zero functions:

Corollary 2.35. If g is polynomial-growth function and $domain(g)$ has a positive lower bound, then there exists a non-negative integer n and a positive real number β such that

$$g(x) \leq \beta x^n.$$

for all $x \in domain(g)$.

Proof. If g is positive, Lemma 2.34 implies the existence of β and n. Suppose g is not positive, so g is identically zero by Lemma 2.7. Since g has a positive domain, the conclusion is satisfied by every choice of n and β. (The assumption that $inf\ domain(g) > 0$ is unnecessary when g is identically zero.) □

Of course, Corollary 2.35 does not include a lower bound for $g(x)$ of the type provided by Lemma 2.34. Identically zero functions on non-empty positive sets have polynomial growth but are not bounded below by αx^k for any combination of $\alpha \in \boldsymbol{R}^+$ and $k \in \boldsymbol{Z}$.

Counterexamples on $(\mathbf{0}, \infty)$. The positive functions x and $1/x$ on $(0, \infty)$ have polynomial growth by Corollary 2.12. There are no choices of $\alpha, \beta \in \boldsymbol{R}^+$ and $m, n \in \boldsymbol{N}$ for which either $\alpha x^{-m} \leq x$ for all $x \in (0, \infty)$ or $1/x \leq \beta x^n$ for all $x \in (0, \infty)$. (Consider the limits as $x \to 0$.) Lemma 2.34 is inapplicable because $(0, \infty)$ has no positive lower bound.

Power bounds do not imply polynomial growth. The obvious converse to Lemma 2.34 for positive functions on real sets with positive lower bounds is false. Define the real-valued function h on $(0, \infty)$ by

$$h(x) = \frac{1}{x} + \frac{1}{2}\left(x - \frac{1}{x}\right)(1 + \sin x).$$

Let $I = [1, \infty)$ and $f = h|_I$, so

$$\frac{1}{x} \leq f(x) \leq x$$

for all $x \in I$. If $k \in \boldsymbol{Z}^+$ and $w = (2k + 1/2)\pi$, then $w \in I$ and $w + \pi \in [w, 2w]$. Observe that $f(w) = w$ and $f(w + \pi) = 1/(w + \pi)$. The function f is positive, so $\Psi_2(f)$ is defined. Furthermore,

$$\Psi_2(f) \geq \frac{f(w)}{f(w + \pi)} = w(w + \pi) > w^2 > 4k^2\pi^2$$

for all k. Therefore, $\Psi_2(f) = \infty$, so Lemma 2.16 implies f does not have polynomial growth.

The positive function f is a counterexample to the converse of Lemma 2.34, but f is not a candidate for Leighton's polynomial-growth condition; the domain of f is incompatible. We now provide two related counterexamples that are candidates for Leighton's polynomial-growth condition relative to some non-empty finite subsets of (0,1) but do not satisfy Leighton's polynomial-growth condition relative to any such subset.

Define continuous real-valued functions p and q on $(0, \infty)$ by $p(x) = x \cdot h(x)$ and $q(x) = h(x)/x$. Continuity of p and q along with $q(x) \leq 1 \leq p(x)$ for all $x \geq 1$ implies there exists $\beta \in (0,1)$ such that $q(x) \leq 2$ and $p(x) \geq 1/2$ for all $x \geq \beta$. Let h^* be the restriction of h to $[\beta, \infty)$, so

$$\frac{1}{2x} \leq h^*(x) \leq 2x$$

for all $x \in [\beta, \infty)$. In particular, h^* is a positive function. Thus h^* is a candidate for Leighton's polynomial-growth condition relative to $\{\beta\}$. Since

$$h^*|_I = \left(h|_{[\beta,\infty)}\right)|_I = h|_I = f$$

does not have polynomial growth, Lemma 2.2(8) implies h^* does not satisfy Leighton's polynomial-growth condition relative to any set.

Now let S be any non-empty, finite subset of the interval $(0,1)$. Let $c = \min S$ and $b = 1/c$, so bx is in the domain, $[1, \infty)$, of f for all x in $[c, \infty)$. We define the function $G: [c, \infty) \to \boldsymbol{R}^+$ by $G(x) = f(bx)$. The function f is positive, so G is also a positive function. Thus G is a candidate for Leighton's polynomial-growth condition relative to the set S. Furthermore,

$$\frac{c}{x} = \frac{1}{bx} \le f(bx) \le bx$$

for all $x \in [c, \infty)$, i.e.,

$$\frac{c}{x} \le G(x) \le bx$$

for all such x. Since

$$Ratios_2(G) = Ratios_2(f),$$

Lemma 2.10(1) implies

$$\Psi_2(G) = \Psi_2(f) = \infty.$$

Lemma 2.16 implies G is not a polynomial-growth function. The function G is continuous, so Lemma 2.14 implies G has polynomial growth on the lower subset $[c, 1]$ of $[c, \infty)$. Since

$$[c, \infty) = [c, 1] \cup [1, \infty),$$

Lemma 2.24 implies G does not have polynomial growth on $[1, \infty)$. By Lemma 2.2(8), the function G does not satisfy Leighton's polynomial-growth condition relative to any set.

3. Non-Polynomial-Growth Functions g With Polynomial-Bounded $|g'(x)|$

In [Le], the statement of Theorem 1 is accompanied by the following remark: "If $|g'(x)|$ is upper bounded by a polynomial in x, then $g(x)$ satisfies the polynomial growth condition." The assertion is incorrect even if we adopt the unstated condition that g is a differentiable candidate for Leighton's polynomial-growth condition. This section supplies four classes of counterexamples: non-constant functions with roots, functions that rapidly approach zero at infinity, functions with large oscillations, and positive increasing functions with long intervals of contrasting growth rates.

From each class of counterexamples, we exhibit a representative non-negative, differentiable, real-valued function g on the positive real numbers. Like every other non-negative function on the positive real numbers, each g is a candidate for Leighton's polynomial-growth condition relative to every non-empty, finite subset of $(0,1)$. However, each representative g fails to satisfy Leighton's polynomial-growth condition relative to any set. Representatives of two classes satisfy $|g'(x)| < x + 1$, and a representative of another satisfies $|g'(x)| < 1$. For all $\varepsilon > 0$, there is a representative of the fourth class with $|g'(x)| < \varepsilon$.

Polynomial upper bound on a candidate. Let g be a differentiable, non-negative, real-valued function on $[\min S, \infty)$ where S is a non-empty finite subset of $(0,1)$. In particular, the function g is a candidate relative to S. Suppose g' is locally Riemann integrable. If $|g'|$ is bounded above by a polynomial, then g' is bounded above by the same polynomial. Integration yields a polynomial upper bound for g. However, as we demonstrated near the end of Section 2, the existence of a polynomial upper bound for a candidate does not imply satisfaction of Leighton's polynomial-growth condition.

Polynomial growth of g does not imply polynomial bound for $|g'|$. Before providing counterexamples to Leighton's remark, we consider the converse. Does satisfaction of Leighton's polynomial-growth condition by a differentiable function g imply $|g'|$ is bounded above by a polynomial? No, it does not. Define $g: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ by

$$g(x) = 2 + sin(e^x),$$

so

$$\Psi_2(g) = 3.$$

Lemma 2.16 implies polynomial growth of g. Corollary 2.17 and Lemma 2.2(2) imply g satisfies Leighton's polynomial-growth condition relative to every non-empty, finite subset of $(0,1)$. However,

$$|g'(x)| = e^x|\cos e^x|.$$

Observe that

$$e^x|\cos e^x| = e^x$$

whenever e^x/π is an integer. Since e^x/π is continuous and approaches ∞ as x approaches ∞, the solutions of

$$|g'(x)| = e^x$$

form an unbounded set. Therefore, $|g'(x)|$ is not bounded above by a polynomial.

Non-constant functions with roots. Suppose $g: \boldsymbol{R}^+ \to \boldsymbol{R}$ is a non-negative function with a root in $[1,\infty)$ such that the restriction of g to $[1,\infty)$ is not identically zero. By Lemma 2.7, the function g does not have polynomial growth on $[1,\infty)$. Lemma 2.2(8) implies g does not satisfy Leighton's polynomial-growth condition relative to any set. There exist many such g that are infinitely differentiable with $|g'(x)|$ bounded above by a polynomial.

For example, define the non-negative function $g: \boldsymbol{R}^+ \to \boldsymbol{R}$ (with unique root 1) by

$$g(x) = \frac{1}{2}(x-1)^2,$$

so

$$g'(x) = x - 1,$$

which combines with $x > 0$ to imply

$$|g'(x)| < x + 1.$$

A related non-counterexample. We now consider a slightly different example (with no roots) that violates our definition of polynomial growth. Lemma 2.19 implies the positive function $g(x) = x - L$ on (L,∞), where L is a positive real number, does not have polynomial growth although $g'(x) = 1$ is a constant polynomial. However, g should not be considered a counterexample to Leighton's remark. If $L \geq 1$, then g is not a candidate for Leighton's polynomial-growth condition. Suppose $L < 1$, so $(L, 1)$ is non-empty. Let S be a non-empty, finite subset of $(L, 1)$, so $\min S > L$. The restriction of g to $[\min S, \infty)$ is positive. Since g is continuous and $g(x) = \Theta(x)$, Corollaries 2.12 and 2.33 implies g has polynomial growth on $[\min S, \infty)$. By Corollary 2.17, the function g satisfies Leighton's polynomial-growth condition relative to S.

Functions that rapidly approach zero at infinity. Define the positive, infinitely differentiable real-valued function $g: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ by $g(x) = e^{-x}$. The reciprocal function $1/g(x) = e^x$ is not bounded above by a polynomial on $[1, \infty)$, so Corollaries 2.15 and 2.35 imply g does not have polynomial growth on $[1, \infty)$. Lemma 2.2(8) implies g does not satisfy Leighton's polynomial-growth condition relative to any set. However,

$$|g'(x)| = e^{-x} < 1$$

for all x in $\boldsymbol{R}^+$, the domain of g. Therefore, $|g'(x)|$ is bounded above by a constant, i.e., a degree zero polynomial, and g is another counterexample.

The following proposition is used later by a couple of our examples.

Lemma 3.1. Suppose $f: [1, \infty) \to \boldsymbol{R}^+$ is continuous and does not have polynomial growth. Define $g: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ by $g(x) = f(x + 1)$. The function g does not satisfy Leighton's polynomial-growth condition relative to any set.

Proof. The interval $[1,2]$ is a lower subset of $domain(f) = [1,2] \cup [2, \infty)$, which is a positive set. The function f is positive and continuous, so the restriction of f to the positive compact interval $[1,2]$ is also positive and continuous. Corollary 2.14 implies f has polynomial growth on $[1,2]$. Let f^* be the restriction of f to $[2, \infty)$. Since f is positive and does not have polynomial growth, we conclude from Lemma 2.24 that f^* does not have polynomial growth.

Let g^* be the restriction of g to $[1, \infty)$. For all $x \in [2, \infty)$, we have

$$[x, 2x] \subseteq [x, 3x - 2] = [(x - 1) + 1, 3(x - 1) + 1].$$

Lemma 2.9(5) implies

$$\Lambda_{f^*}([x, 2x]) \leq \Lambda_{f^*}([x, 3x - 2]) = \Lambda_{g^*}([(x - 1), 3(x - 1)]) \leq \Psi_3(g^*),$$

so

$$\Psi_2(f^*) \leq \Psi_3(g^*).$$

Lemma 2.16 implies

$$\Psi_2(f^*) = \infty,$$

so

$$\Psi_3(g^*) = \infty.$$

Lemma 2.16 implies g^* does not have polynomial growth. Lemma 2.2(8) implies g does not satisfy Leighton's polynomial-growth condition relative to any set. □

Of course, the function g of Lemma 3.1 is a candidate for Leighton's polynomial-growth condition relative to each non-empty, finite subset of $(0,1)$.

Domain, positivity, and continuity in Lemma 3.1. Define $f: [1, \infty) \to \boldsymbol{R}$ and $g: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ by $f(x) = x - 1$ and $g(x) = f(x+1) = x$. Since 1 is the unique root of f, the function f is neither positive nor identically zero. Lemma 2.7 implies f does not have polynomial growth. However, the function g has polynomial growth by Corollary 2.12. Lemma 2.2(2) and Corollary 2.17 imply g satisfies Leighton's polynomial-growth condition relative to every non-empty, finite subset of $(0,1)$. Although f is continuous, Lemma 3.1 is inapplicable because f is not a positive function.

What happens if we delete the root, 1, from the domain of f? Define $f: (1, \infty) \to \boldsymbol{R}^+$ and $g: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ by $f(x) = x - 1$ and $g(x) = f(x+1) = x$. As explained earlier, Lemma 2.19 implies f does not have polynomial growth. As before, g has polynomial growth by Corollary 2.12 and satisfies Leighton's polynomial-growth condition relative to every non-empty, finite subset of $(0,1)$. Although f is positive and continuous, Lemma 3.1 is inapplicable because the domain of f is $(1, \infty)$ instead of $[1, \infty)$. (Lemma 3.1 remains true if $domain(f) = [c, \infty)$ and $g(x) = f(x+c)$ for some $c > 0$.)

Now instead define $f: [1, \infty) \to \boldsymbol{R}^+$ by $f(1) = 1$ and $f(x) = 1/(x-1)$, so $f(x)$ approaches ∞ as x approaches 1. Lemma 2.19 implies f does not have polynomial growth. As before, define $g: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ by $g(x) = f(x+1)$, so $g(x) = 1/x$. Corollary 2.12 implies g has polynomial growth. Lemma 2.2(2) and Corollary 2.17 imply g satisfies Leighton's polynomial-growth condition relative to every non-empty, finite subset of $(0,1)$. Although f is positive, Lemma 3.1 is inapplicable because f is not continuous at 1.

Functions with large oscillations. Near the end of Section 2, we a defined positive, infinitely differentiable functions $f: [1, \infty) \to \boldsymbol{R}^+$ by

$$f(x) = \frac{1}{x} + \frac{1}{2}\left(x - \frac{1}{x}\right)(1 + \sin x)$$

and showed that f does not have polynomial growth. Define the positive, infinitely differentiable function $g: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ by

$$g(x) = f(x+1).$$

Lemma 3.1 implies g does not satisfy Leighton's polynomial-growth condition relative to any set. We claim

$$|f'(x)| < x$$

for all $x \in [1, \infty)$, so

$$|g'(x)| = |f'(x+1)| < x + 1$$

for all $x \in \boldsymbol{R}^+$. The derivative of f is

$$f'(x) = \frac{-1}{x^2} + \frac{1}{2}\left(1 + \frac{1}{x^2}\right)(1 + \sin x) + \frac{1}{2}\left(x - \frac{1}{x}\right)\cos x.$$

Observe that $f'(1) = \sin 1$ and $0 < 1 < \frac{\pi}{2}$, so $0 < \sin 1 < 1$, i.e, $|f'(1)| = f'(1) < 1$. Now suppose $x > 1$, so $x > 1/x$. At most one of $\sin x$ and $\cos x$ equals 1, so

$$f'(x) < 1 + \frac{1}{2}\left(x - \frac{1}{x}\right) = \frac{x^2 + 2x - 1}{2x} = x - \frac{(x-1)^2}{2x} < x.$$

At most one of $\sin x$ and $\cos x$ equals -1, so

$$f'(x) + x > \frac{-1}{x^2} + \frac{1}{2}\left(\frac{1}{x} - x\right) + x = \frac{x^3 + x - 2}{2x^2} > 0,$$

i.e., $f'(x) > -x$. Therefore, $|f'(x)| < x$ as claimed, and g is a counterexample to Leighton's remark.

Positive, increasing functions with long intervals of contrasting growth rates. We claim that for all $\varepsilon > 0$, there exists a corresponding positive, increasing, continuously differentiable function $g: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ that satisfies $|g'(x)| = g'(x) < \varepsilon$ for all $x \in \boldsymbol{R}^+$ but does not satisfy Leighton's polynomial-growth condition relative to any set. Observe that ε is a zero-degree polynomial upper bound for $|g'(x)|$, so g is another counterexample.

Let $\varepsilon > 0$. We construct g by splicing together linear and quadratic polynomial functions defined on intervals. The quadratic segments will be defined as follows: If a, b, c, and d are real numbers, then the function $f: [c, c+1] \to \boldsymbol{R}$ defined by

$$f(x) = \frac{b-a}{2}(x-c)^2 + a(x-c) + d$$

is the unique polynomial function on $[c, c+1]$ with real coefficients and degree at most two that satisfies $f'(c) = a$, $f'(c+1) = b$, and $f(c) = d$. The function f is of course continuously differentiable. The derivative f' is monotonic because its degree is at most one, so

$$\min(a, b) \le f'(x) \le \max(a, b)$$

for all $x \in [c, c+1]$. If a and b are positive, then f' is positive and f is increasing; if d is also positive, then f is positive. Observe that

$$f(c+1) = d + \frac{(a+b)}{2} \le \mathrm{d} + \max(a, b).$$

We inductively define an increasing sequence $x_0, x_1, x_2, \ldots$ of positive real numbers by $x_0 = 1$ and

$$x_{n+1} = e^{2x_n + 1}$$

for each non-negative integer n. Observe that $x - \log x - 1$ has the root 1 and has a positive derivative on $(1, \infty)$, which implies $x - \log x > 1$ for all $x > 1$. Positivity of x_n implies $x_{n+1} > 1$, so

$$x_{n+1} > \log x_{n+1} + 1 = 2x_n + 2.$$

Let $\delta = \min(\varepsilon/2\,, 1/2)$. Another sequence $\beta_0, \beta_1, \beta_2, \ldots$ of positive real numbers is defined by

$$\beta_n = \frac{\delta}{x_{n+1} - \log x_{n+1}},$$

so $\beta_n < \delta$ and

$$\beta_n \le \frac{1}{2(x_{n+1} - \log x_{n+1})} < \frac{1}{2(x_{n+1} - \log x_{n+1} - 1)}.$$

Define a sequence of intervals $I_0, I_1, I_2, \ldots$ in $(0, \infty)$ by

$$I_{4n} = \begin{cases} (0, x_0], & \text{for } n = 0 \\ [x_n - 1, x_n], & \text{for } n > 0, \end{cases}$$

$$I_{4n+1} = [x_n, 2x_n],$$

$$I_{4n+2} = [2x_n, 2x_n + 1] = [2x_n, \log x_{n+1}],$$

and

$$I_{4n+3} = [2x_n + 1, x_{n+1} - 1] = [\log x_{n+1}\,, x_{n+1} - 1]$$

for each non-negative integer n, so

$$(0, \infty) = \bigcup_{k=0}^{\infty} I_k.$$

Each interval I_k has positive length, and $\max I_k = \min I_{k+1}$. We now recursively define a sequence $h_0, h_1, h_2, \ldots$ of positive, increasing, continuously differentiable functions with $h_k : I_k \to \boldsymbol{R}^+$ for each non-negative integer k. Let n be a non-negative integer. If $n = 0$, then $h_{4n}(x) = h_0(x) = \delta x$ for all $x \in I_0$; otherwise, let h_{4n} be the quadratic function determined by $h'_{4n}(x_n - 1) = \beta_{n-1}$, $h'_{4n}(x_n) = \delta$, and

$$h_{4n}(x_n - 1) = h_{4n-1}(x_n - 1).$$

Define h_{4n+1} by

$$h_{4n+1}(x) = h_{4n}(x_n) + \delta(x - x_n).$$

The function h_{4n+2} is the quadratic function determined by $h'_{4n+2}(2x_n) = \delta$, $h'_{4n+2}(2x_n + 1) = \beta_n$, and

$$h_{4n+2}(2x_n) = h_{4n+1}(2x_n).$$

The function h_{4n+3} is defined by

$$h_{4n+3}(x) = h_{4n+2}(2x_n + 1) + \beta_n(x - 2x_n - 1).$$

Since

$$h_k(\max I_k) = h_{k+1}(\max I_k)$$

for each non-negative integer k, there exists exactly one function g on $(0, \infty)$ such that $g|_{I_k} = h_k$ for all such k. The function g is positive and increasing because each h_k is positive and increasing. Furthermore, g is continuously differentiable because each h_k is continuously differentiable with

$$h'_k(\max I_k) = h'_{k+1}(\max I_k).$$

The derivative g' is a positive function, and each derivative h'_k is monotonic, so

$$\sup_{x\in(0,\infty)} |g'(x)| = \sup_{x\in(0,\infty)} g'(x) = \sup_{k\geq 0}\left(\sup_{x\in I_k}\big(h'_k(x)\big)\right) = \max\left(\delta, \sup_{k\geq 0} \beta_k\right) = \delta < \varepsilon.$$

Since $g(t) = \delta t$ for all $t \in (0, 1]$, we conclude that $g(x) \leq \delta x$ for all $x \in (0, \infty)$.
If $n > 0$, then

$$g(x_n - 1) = g(\log x_n) + \beta_{n-1}(x_n - \log x_n - 1) < \delta \log x_n + 1/2,$$

$$g(x_n) \leq g(x_n - 1) + \delta < \delta \log x_n + 1,$$

and

$$\frac{g(2x_n)}{g(x_n)} = \frac{g(x_n) + \delta x_n}{g(x_n)} > \frac{\delta x_n}{\delta \log x_n + 1}.$$

It follows from

$$\lim_{n\to\infty} x_n = \infty$$

and

$$\lim_{t\to\infty} \frac{\delta t}{\delta \log t + 1} = \infty$$

that

$$\lim_{n\to\infty} \frac{g(2x_n)}{g(x_n)} = \infty.$$

The interval $[1, \infty)$ contains each x_n, so

$$\Psi_2\big(g|_{[1,\infty)}\big) = \infty.$$

Lemma 2.16 implies g does not have polynomial growth on $[1, \infty)$. By Lemma 2.2(8), the function g does not satisfy Leighton's polynomial-growth condition relative to any non-empty finite subset of (0,1).

A positive, increasing counterexample based on the error function. Just for fun, we construct another counterexample to Leighton's remark about derivatives and his polynomial-growth condition. In Theorem 3.6, we shall construct a positive, increasing, continuously differentiable, real-valued function f on $[1,\infty)$ that satisfies

$$|f'(x)| = f'(x) < x$$

for all $x \in [1,\infty)$ but does not have polynomial growth.

Define a positive, increasing, continuously differentiable function $g\colon \boldsymbol{R}^+ \to \boldsymbol{R}^+$ by $g(x) = f(x+1)$. Lemma 3.1 implies g does not satisfy Leighton's polynomial-growth condition relative to any set. Observe that

$$|g'(x)| = g'(x) = f'(x+1) < x+1$$

for all x in the domain of g.

***P* and *E*.** Unlike the earlier positive, increasing counterexample that is pieced together from linear and quadratic polynomials, the functions f and g are based on the error function, which is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

for each real number x. Define the functions

$$P(x,y) = \frac{y}{4}e^{-(x-y)^2}$$

and

$$E(x,y) = \frac{\sqrt{\pi}}{8}y\cdot(\operatorname{erf}(x-y)+1)$$

on the real plane. They are related by

$$\frac{\partial E}{\partial x} = P.$$

If $y > 0$, then

$$0 < P(x,y) \le \frac{y}{4}$$

and

$$0 < E(x,y) < \frac{\sqrt{\pi}}{4}y < \frac{y}{2}$$

for all x. The equality $P(x,y) = \frac{y}{4}$ holds if and only if $x = y$. Likewise, $E(x,y) = \frac{\sqrt{\pi}}{8}y$ if and only if $x = y$. For each real number y, we have

$$\lim_{x\to\infty} P(x,y) = 0.$$

For each real number x, we have

$$\lim_{y\to\infty} P(x,y) = 0$$

and

$$\lim_{y\to\infty} E(x,y) = 0.$$

Furthermore,

$$\lim_{x\to\infty} E(x,2x) = 0.$$

The second and third limits above follow from L'Hôpital's rule, and the fourth limit follows from the third: For all real x, we have

$$\lim_{y\to\infty} \frac{y}{e^{(x-y)^2}} = \lim_{y\to\infty} \frac{1}{2(y-x)e^{(x-y)^2}} = 0,$$

which implies

$$\lim_{y\to\infty} P(x,y) = 0.$$

Similarly,

$$4\cdot \lim_{y\to\infty} E(x,y) = \frac{\sqrt{\pi}}{2}\cdot \lim_{y\to\infty} \frac{\operatorname{erf}(x-y)+1}{y^{-1}} = \lim_{y\to\infty} \frac{y^2}{e^{(x-y)^2}}$$

$$= \lim_{y\to\infty} \frac{y}{(y-x)e^{(x-y)^2}} = \left(\lim_{y\to\infty} \frac{1}{(y-x)}\right)\left(\lim_{y\to\infty} \frac{y}{e^{(x-y)^2}}\right) = 0\cdot 0 = 0,$$

so

$$\lim_{y\to\infty} E(x,y) = 0.$$

Finally,

$$\lim_{x\to\infty} E(x,2x) = 2\cdot \lim_{x\to\infty} E(0,x) = 0.$$

Lemma 3.2. $P(x,y) < x - \frac{1}{2}$ for all x, y in $[1,\infty)$.

Proof. Let $x \geq 1$, and define the function $h(y) = P(x,y)$ on $[1,\infty)$, so that

$$h'(y) = \frac{1}{4}e^{-(x-y)^2}(2y(x-y)+1).$$

Since

$$\frac{1}{4}e^{-(x-y)^2} > 0$$

for all real y, a real number u is a critical point of h if and only if u is contained in $domain(h)$, and u is one of the roots

$$w = \frac{1}{2}\left(x + \sqrt{x^2+2}\right)$$

or

$$w^* = \frac{1}{2}\left(x - \sqrt{x^2+2}\right)$$

of the quadratic equation

$$2y(x-y)+1=0.$$

Observe that

$$1 \le x < w < x+1.$$

In particular, $w \in domain(h)$, whereas $w^* < 0$, which implies $w^* \notin domain(h)$. Thus w is the unique critical point of h. Since $h(1) > 0$ and

$$\lim_{y\to\infty} h(y) = 0,$$

there exists $t > 1$ with $h(u) < h(1)$ for all $u \ge t$. Continuity of h implies there exists $m \in [1,t]$ such that $h(m) \ge h(z)$ for all $z \in [1,t]$. Exactly one of the following conditions is satisfied for any such m:

(1) m is a critical point of h, i.e., $m = w$.
(2) $m = 1$ and $h'(1) < 0$.
(3) $m = t$ and $h'(t) > 0$.

Condition (2) is violated because $h'(1) > 0$. The inequalities $h(t) < h(1) \le h(m)$ imply $m \ne t$, i.e., condition (3) is violated. Therefore (1) is satisfied, i.e., $m = w$. Furthermore,

$$h(w) > h(1) > h(u)$$

for all $u \ge t$. We conclude that $h(w)$ is the maximum value of the function h. For all y in $[1,\infty)$, we have

$$P(x,y) \;\le\; P(x,w) < \frac{w}{4} < \frac{x+1}{4} \;\le\; x - \frac{1}{2}.$$

□

Definition. For each real number $y \ge 1$ and each positive integer n, let

$$A(y,n) = B(y,n) \cup C(y,n)$$

where

$$B(y,n) = \left\{x \in [1,\infty)\colon\; P(x,y) \ge \frac{1}{2^{n+1}}\right\}$$

and

$$C(y,n) = \left\{x \in [1,y]\colon\; E(x,y) \ge \frac{1}{2^{n+2}}\right\}.$$

Lemma 3.3. If $y \ge 1$ is a real number and n is a positive integer, then $A(y,n)$ is a closed and bounded subinterval of $[1,\infty)$ containing y.

Proof. Let $A = A(y,n)$, $B = B(y,n)$, and $C = C(y,n)$. By definition, B is contained in $[1,\infty)$, and C is contained in $[1,y]$, which is contained in $[1,\infty)$. Therefore, the union A of B and C is also contained in $[1,\infty)$. The relations

$$P(y,y) = \frac{y}{4} \geq \frac{1}{4} \geq \frac{1}{2^{n+1}} \quad \text{and} \quad E(y,y) = \frac{\sqrt{\pi}}{8} y > \frac{1}{8} \geq \frac{1}{2^{n+2}}$$

imply y is an element of B and C and is therefore also an element of $A = B \cup C$.

Define the functions $P_y: \boldsymbol{R} \to \boldsymbol{R}$ and $E_y: \boldsymbol{R} \to \boldsymbol{R}$ by $P_y(x) = P(x,y)$ and $E_y(x) = E(x,y)$. The set B is the intersection of the closed set $[1,\infty)$ and the closed preimage of the closed set $[1/2^{n+1}, \infty)$ under the continuous function P_y. The set C is the intersection of the closed interval $[1,y]$ and the closed preimage of the closed set $[1/2^{n+2}, \infty)$ under the continuous function E_y. Therefore, B, C, and $A = B \cup C$ are closed subsets of the real numbers.

The sets C and $B \cap [1,y]$ are connected because E_y and the restriction of P_y to $[1,y]$ are increasing. The set $B \cap [y,\infty)$ is connected because the restriction of P_y to $[y,\infty)$ is decreasing. B is connected because it is the union of non-disjoint connected sets $B \cap [1,y]$ and $B \cap [y,\infty)$. (They both contain y.) The set A is connected because it is the union of non-disjoint connected sets B and C. (They both contain y.) In other words, A is an interval.

By definition, the set B is bounded below by 1. B is bounded above because

$$\lim_{x\to\infty} P_y(x) = 0.$$

C is bounded because it is contained in the bounded interval $[1,y]$. A is bounded because it is the union of the bounded sets B and C. □

It can be easily shown that $A(y,n)$ is the degenerate interval $[y,y]$ if and only if $y = n = 1$. However, we shall not need this fact.

Lemma 3.4. For each real number $c \geq 1$ and each positive integer n, there exists a real number $w > c$ such that $A(c,n)$ and $A(d,n+1)$ are disjoint for each real number d satisfying $d \geq w$.

Proof. Lemma 3.3 implies $A(c,n)$ has a maximum element u. The limits

$$\lim_{y\to\infty} P(u,y) = 0 \text{ and } \lim_{y\to\infty} E(u,y) = 0$$

imply there exists $w > u$ such that $P(u,d) < \frac{1}{2^{n+2}}$ and $E(u,d) < \frac{1}{2^{n+3}}$ for all $d \geq w$, so $u \notin A(d,n+1)$ for all such d. By Lemma 3.3, $A(d,n+1)$ is a closed and bounded interval containing d. The conditions $u \notin A(d,n+1)$ and

$$u < w \leq d \in A(d,n+1)$$

imply $u < min\big(A(d, n+1)\big)$. Therefore, $A(c, n)$ and $A(d, n+1)$ are disjoint. □

Lemma 3.5. There exists an infinite sequence $a_1, a_2, a_3, \ldots$ in $[1, \infty)$ such that

$$a_{n+1} \geq \left(\sum_{i=1}^{n} a_i\right)^2 + 1$$

and

$$\max\big(A(a_n, n)\big) < \min\big(A(a_{n+1}, n+1)\big)$$

for all $n \in \boldsymbol{Z}^+$.

Proof. We provide an inductive definition of such a sequence. Let a_1 be any element of $[1, \infty)$. Given a positive integer n, and the partial sequence $a_1 \ldots a_n$, Lemma 3.4 implies there exists $w > a_n$ such that $A(a_n, n)$ and $A(d, n+1)$ are disjoint for each real number $d \geq w$.

Define

$$a_{n+1} = \max\left(w, \left(\sum_{i=1}^{n} a_i\right)^2 + 1\right),$$

so a_{n+1} satisfies the first required inequality. Furthermore, $A(a_n, n)$ and $A(a_{n+1}, n+1)$ are disjoint. By Lemma 3.3, $A(a_n, n)$ and $A(a_{n+1}, n+1)$ are closed and bounded intervals containing a_n and a_{n+1} respectively. We conclude from $a_n < a_{n+1}$ that

$$\max\big(A(a_n, n)\big) < \min\big(A(a_{n+1}, n+1)\big).$$

□

We are now ready to construct the promised function f:

Theorem 3.6. Let $a_1, a_2, a_3, \ldots$ be as in Lemma 3.5, and define a sequence $f_1, f_2, f_3, \ldots$ of real-valued functions on $[1, \infty)$ by $f_n(x) = E(x, a_n)$. The series $\sum f_n$ converges pointwise to a positive, increasing, continuously differentiable function f on $[1, \infty)$ such that $0 < f'(x) < x$ for all $x \in [1, \infty)$. The function f does not have polynomial growth.

Proof. Define functions $p_1, p_2, p_3, \ldots$ on $[1, \infty)$ by $p_n(t) = P(t, a_n)$. Let $x \in [1, \infty)$. The functions f_n and p_n are positive, so the series $\sum f_n(x)$ and $\sum p_n(x)$ either converge to positive real numbers, or diverge to $+\infty$.

Define $A_n = A(a_n, n)$ for each positive integer n. By Lemma 3.3, A_n is a closed and bounded subinterval of $[1, \infty)$ containing a_n. Since the increasing sequence $a_1, a_2, a_3, \ldots$ approaches infinity and $a_n \leq \max A_n < \min A_{n+1}$, the increasing sequence

$$\min A_1, \min A_2, \min A_3, \ldots$$

also approaches infinity.

Let k be the least non-negative integer for which $x < \min\ A_{k+1}$, so x lies outside A_n for each positive integer $n \neq k$. For all $n > k$, we have $x < \min(A_n) \leq a_n$, which combines with $x \notin C(a_n, n) \subseteq A_n$ to imply $f_n(x) < 1/2^{n+2}$. For each positive integer n, the inequality $f_n(x) < a_n/2$ holds. If $k = 0$, then

$$\sum_{n=1}^{\infty} f_n(x) \ < \ \sum_{n=1}^{\infty} \frac{1}{2^{n+2}} \ = \ \frac{1}{4} \ < \ \infty.$$

If $k > 0$, then

$$\sum_{n=1}^{\infty} f_n(x) \ < \ \sum_{n=1}^{k} \frac{a_n}{2} + \sum_{n=k+1}^{\infty} \frac{1}{2^{n+2}} \ < \ \frac{\sqrt{a_{k+1}}}{2} + \frac{1}{8} \ < \ \infty.$$

Hence $\sum f_n$ converges pointwise to a positive real-valued function f.

If $k \leq 1$, define $v = 1$; otherwise,

$$\max(A_{k-1}) < \ \min(A_k) \leq x,$$

and we define

$$v = \frac{\max(A_{k-1}) + \ \min(A_k)}{2}.$$

Let

$$w = \frac{x + \min(A_{k+1})}{2},$$

and define the closed interval $J = [v, w]$, which is disjoint from A_n for each positive integer $n \neq k$. Observe that J has positive length, and $x \in J \subseteq [1, \infty)$. Furthermore, x is in the interior of J if $x \neq 1$.

Let z be any element of J. If $k = 0$, then

$$\sum_{n=1}^{\infty} p_n(z) < \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2} < z.$$

If $k > 0$, it follows from Lemma 3.2 that

$$\sum_{n=1}^{\infty} p_n\,(z) \ < \ p_k(z) + \sum_{n \neq k} \frac{1}{2^{n+1}} \ < \ \left(z - \frac{1}{2}\right) + \frac{1}{2} \ = \ z.$$

In particular, $\sum p_n(x)$ converges to a real number less than x. Hence $\sum p_n$ converges pointwise to a positive function p, and $p(x) < x$. Convergence is uniform on J since

$$\sum_{n=m}^{\infty} p_n(z) \;<\; \sum_{n=m}^{\infty} \frac{1}{2^{n+1}} \;=\; \frac{1}{2^m}$$

for each integer $m > k$, and 2^{-m} approaches zero as m approaches infinity. (The series does not converge uniformly on $[1, \infty)$.)

Uniform convergence of $\sum p_n$ on J combines with continuity of each p_n to imply continuity of $p|_J$. In particular, $p|_J$ is continuous at x. Either x is in the interior of J, or $x = 1$. Recall that J has positive length. Hence J contains the intersection of $[1, \infty)$ with some open interval containing x. (The open interval is not contained in $[1, \infty)$ if $x = 1$.) Therefore, continuity of p at x follows from continuity of $p|_J$ at x. Thus p is continuous.

The identity $\frac{\partial E}{\partial x} = P$ implies the derivative of f_n is p_n for all n. Pointwise convergence of $\sum f_n$ to f combines with the uniform convergence of $\sum p_n$ on J to imply that the restriction of f to J is differentiable, and its derivative is the restriction of p to J. See, for example, Theorem 7.17 of [Ru]. If $x \neq 1$, then x is an interior point of J, so that f is differentiable at x, and $f'(x) = p(x)$. If $x = 1$, then $v = 1$, and the restriction of f to J has a one sided derivative at 1 given by

$$\left(f|_J\right)'(1) = p(1);$$

furthermore, f has a one sided derivative at 1, and $f'(1) = p(1)$. Thus $f' = p$, which implies

$$0 < f'(x) < x.$$

Since p is positive and continuous, the function f is increasing and continuously differentiable.

If $n > 1$ is an integer, then $a_n \geq a_2 \geq {a_1}^2 + 1 \geq 2$,

$$f(a_n) > f_n(a_n) = E(a_n, a_n) = \frac{\sqrt{\pi}}{8} a_n,$$

$$1 \leq \frac{a_n}{2} < \min A_{n+1},$$

and

$$f\left(\frac{a_n}{2}\right) \;<\; \sum_{i=1}^{n-1} \frac{a_i}{2} + E\left(\frac{a_n}{2}, a_n\right) + \sum_{i=n+1}^{\infty} \frac{1}{2^{i+2}} \;<\; \frac{\sqrt{a_n}}{2} + E\left(\frac{a_n}{2}, a_n\right) + \frac{1}{16}.$$

It follows from

$$\lim_{n\to\infty} a_n = \infty \text{ and } \lim_{n\to\infty} E\left(\frac{a_n}{2}, a_n\right) = 0$$

that

$$\lim_{n\to\infty}\frac{f(a_n)}{f(a_n/2)} \geq \lim_{n\to\infty}\left(\frac{\frac{\sqrt{\pi}}{8}a_n}{\frac{\sqrt{a_n}}{2}+E\left(\frac{a_n}{2},a_n\right)+\frac{1}{16}}\right) = \lim_{n\to\infty}\frac{\sqrt{\pi a_n}}{4} = \infty.$$

Therefore, $\Psi_2(f) = \infty$. Lemma 2.16 implies f does not have polynomial growth. □

The function f of Theorem 3.6 has a positive derivative, so

$$|f'(x)| = f'(x) < x$$

for all $x \in [1, \infty)$ as claimed.

4. Common Polynomial-Growth Functions

In this section, we describe how to recognize many polynomial-growth functions described by formulas, including the (properly interpreted) correct examples in [Le] if we take into account Lemma 2.32. We start with some important special cases:

Lemma 4.1. If D is a positive set, the following functions have polynomial growth on D:

(1) non-negative constant functions.
(2) x^α for each real exponent α.
(3) $\log_b x$ for $b > 1$ if and only if $\inf D > 1$ or $D = \{1\}$.
(4) $\text{floor}(x)$ if and only if $D \subseteq [1, \infty)$ or $D \subseteq (0,1)$.
(5) $\text{ceiling}(x)$.

Proof. Parts (1) and (2) are merely repetitions of Lemma 2.3 and Corollary 2.12 respectively. They are included here for convenience.

Let $c = \inf D$. Suppose $\log_b x$ has polynomial growth on D, and $D \neq \{1\}$. Lemma 2.7 implies $D \subseteq (1, \infty)$, and Lemma 2.19 implies 1 is not a limit point of D. Therefore, $c > 1$.

We now consider the converse portion of (3). The function $\log_b x$ is identically zero on $\{1\}$ and therefore has polynomial growth on $\{1\}$ by Lemma 2.3. Now suppose $c > 1$. The empty function has polynomial growth, so we may assume $D \neq \emptyset$ and $c < \infty$. Define $I = [c, \infty)$, so $D \subseteq I$. Let f be the restriction of $\log_b x$ to the interval I. The function f is increasing, so

$$\Lambda_f([x, 2x]) = \frac{f(2x)}{f(x)}$$

for all x in I. Since $f(2x)/f(x)$ is a decreasing function, we conclude that

$$\Psi_2\big((\log_b x)|_I\big) = \frac{f(2c)}{f(c)} < \infty.$$

Lemma 2.16 implies f has polynomial growth, i.e., $\log_b x$ has polynomial growth on I. Since $D \subseteq I$, Lemma 2.2(2) implies $\log_b x$ has polynomial growth on D. Part (3) is proved.

Lemma 2.16 implies $\mathrm{floor}(x)$ has polynomial growth on $[1, \infty)$ because

$$\Psi_2\big(\mathrm{floor}|_{[1,\infty)}\big) = 3 < \infty.$$

The restriction of $\mathrm{floor}(x)$ to the interval $(0,1)$ is identically zero and has polynomial growth by Lemma 2.3. Lemma 2.2(2) implies $\mathrm{floor}(x)$ has polynomial growth on D when $D \subseteq [1, \infty)$ or $D \subseteq (0,1)$.

Conversely, suppose $\mathrm{floor}(x)$ has polynomial growth on D. By Lemma 2.7, $\mathrm{floor}(x)$ is either positive on D or identically zero on D. Therefore, $D \subseteq [1, \infty)$ or $D \subseteq (0,1)$. Part (4) is proved.

Lemma 2.16 implies the ceiling function has polynomial growth on $(0, \infty)$ because

$$\Psi_2\big(\mathrm{ceiling}|_{(0,\infty)}\big) = 2 < \infty.$$

Lemma 2.2(2) implies the ceiling function has polynomial growth on D. □

Many polynomial-growth functions of interest, including all examples in [Le], are a mixture of some of the ingredients listed in Lemma 2.1: constants, powers, logarithms, floors, and ceilings. It is often possible to instantly recognize polynomial growth of such combinations.

Lemma 4.2. If f and g are positive functions on a positive set D, then

$$\Psi_b(f+g) \le \Psi_b(f) + \Psi_b(g)$$

and

$$\Psi_b(f \cdot g) \le \Psi_b(f) \cdot \Psi_b(g)$$

for all $b > 1$.

Proof. Elements of $\mathrm{Ratios}_b(f+g)$ and $\mathrm{Ratios}_b(f \cdot g)$ are of the form

$$\frac{(f+g)(y)}{(f+g)(x)} = \frac{f(y)}{f(x)+g(x)} + \frac{g(y)}{f(x)+g(x)} < \frac{f(y)}{f(x)} + \frac{g(y)}{g(x)}$$

and

$$\frac{(f \cdot g)(y)}{(f \cdot g)(x)} = \frac{f(y)}{f(x)} \cdot \frac{g(y)}{g(x)},$$

respectively, where $x, y \in D$ such that $x \le by$ and $y \le bx$. Therefore,

$$\sup \text{Ratios}_b(f+g) \leq \sup \text{Ratios}_b(f) + \sup \text{Ratios}_b(g)$$

and

$$\sup \text{Ratios}_b(f \cdot g) \leq \sup \text{Ratios}_b(f) \cdot \sup \text{Ratios}_b(g).$$

By Lemma 2.10(1),

$$\Psi_b(f+g) \leq \Psi_b(f) + \Psi_b(g)$$

and

$$\Psi_b(f \cdot g) \leq \Psi_b(f) \cdot \Psi_b(g).$$

□

Corollary 4.3. If f and g are polynomial-growth functions on a positive set D, then the sum $f+g$ and product $f \cdot g$ have polynomial growth.

Proof. If f is identically zero, then $f+g=g$ and $f \cdot g = f$, and the result follows from polynomial growth of f and g. Therefore, we may assume f is not identically zero. We may similarly assume g is not identically zero.

Corollary 2.30 implies f and g can be extended to polynomial-growth functions F and G, respectively, on $\boldsymbol{R}^+$. Neither F nor G is identically zero, so F and G are positive by Lemma 2.7.

For $b > 1$, Lemmas 4.2 and 2.16 imply

$$\psi_b(F+G) \leq \psi_b(F) + \psi_b(G) < \infty$$

and

$$\psi_b(F \cdot G) \leq \psi_b(F) \cdot \psi_b(G) < \infty.$$

It follows from Lemma 2.16 that $F+G$ and $F \cdot G$ have polynomial growth. Polynomial growth of $f+g$ and $f \cdot g$ follows from Lemma 2.2(2). □

Lemma 4.1 and Corollary 4.3 imply that all polynomial functions with non-negative coefficients have polynomial growth on $(0, \infty)$. As we shall see, there exist polynomial functions with some negative coefficients that also have polynomial growth on $(0, \infty)$. A simple criterion for polynomial growth of a polynomial function is provided later in this section.

Corollary 4.4. If f and g are polynomial-growth functions with the same domain, and g is positive, then f/g has polynomial growth.

Proof. Since $f/g = f \cdot (1/g)$, Corollaries 2.15 and 4.3 imply f/g has polynomial growth. □

Subtraction of polynomial-growth functions. Unlike addition, multiplication, and division, subtraction does not always preserve polynomial growth. For example the function $x - x^2$ on $(1, \infty)$ violates the non-negativity requirement of a polynomial-growth function. (See Lemma 2.2(1).) Furthermore, positivity of a difference does not guarantee polynomial growth. The function $x^2 - x$ on $(1, \infty)$ is positive, but Lemma 2.19 and

$$\lim_{x \to 1^+} (x^2 - x) = 0$$

imply it does not have polynomial growth. Another example is given by the polynomial-growth functions $a(x) = x + 1$ and $b(x) = x + 1 - e^{-x}$ on $[1, \infty)$. (The function $1 - e^{-x}$ on $[1, \infty)$ has polynomial growth by Corollary 2.13, so $b(x)$ has polynomial growth by Corollary 4.3.) The difference $a(x) - b(x) = e^{-x}$ is positive but does not have polynomial growth (see Corollaries 2.15 and 2.35).

However, a polynomial-growth function is obtained if a small enough function (that need not have polynomial growth) is subtracted from (or added to) a positive polynomial-growth function:

Lemma 4.5. If f is a positive polynomial-growth function with domain D, and g is a real-valued function on D such that

$$\sup_{x \in D} \frac{|g(x)|}{f(x)} < 1,$$

then $f + g$ and $f - g$ are polynomial-growth functions.

Proof. We may assume D is non-empty since the empty function has polynomial growth. Corollary 2.30 implies f can be extended to a polynomial-growth function F on $\boldsymbol{R}^+$. Since f is positive and D is non-empty, the function F is not identically zero. Lemma 2.7 implies F is positive.

Let $b > 1$ and

$$c = \sup_{x \in \boldsymbol{D}} \frac{|g(x)|}{f(x)},$$

so $0 \leq c < 1$. Define $G: \boldsymbol{R}^+ \to \boldsymbol{R}$ by $G|_D = g$ and $G(t) = cF(t)$ for all $t \notin D$. Then

$$\sup_{u \in \boldsymbol{R}^+} \frac{|G(u)|}{F(u)} = c,$$

so

$$0 < (1 - c)F(u) \leq F(u) + G(u) \leq (1 + c)F(u)$$

for all $u \in \boldsymbol{R}^+$. In particular, the function $F + G$ is positive. Given $z \in \text{Ratios}_b(F + G)$, there exists $x, y \in \boldsymbol{R}^+$ such that $x \leq by$, $y \leq bx$, and

$$z = \frac{(F + G)(y)}{(F + G)(x)} \leq \left(\frac{1 + c}{1 - c}\right) \frac{F(y)}{F(x)}.$$

It follows from $F(y)/F(x) \in \text{Ratios}_b(F)$ and Lemma 2.10(1) that

$$\frac{F(y)}{F(x)} \leq \sup \text{Ratios}_b(F) = \Psi_b(F)$$

and

$$\Psi_b(F+G) = \sup \text{Ratios}_b(F+G) \leq \left(\frac{1+c}{1-c}\right)\Psi_b(F).$$

Lemma 2.16 implies $\Psi_b(F)$ is finite, so $\Psi_b(f+g)$ is finite. By Lemma 2.16, $F+G$ has polynomial growth. Lemma 2.2(2) implies $f+g$ has polynomial growth. Since $|-g(x)| = |g(x)|$, we conclude that $f+(-g)$ also has polynomial growth. In other words, $f-g$ has polynomial growth. □

Relationship of Lemma 4.5 to Lemma 2.32. When D has a positive lower bound and no finite upper bound, Lemma 4.5 is a special case of Lemma 2.32: The restriction on $|g|/f$ implies $f+g = \Theta(f)$ and $f-g = \Theta(f)$. Corollary 2.22 implies f is locally $\Theta(1)$, which combines with the restriction on $|g|/f$ to imply $f+g$ and $f-g$ are also locally $\Theta(1)$. Therefore, $f+g$ and $f-g$ have polynomial growth by Lemma 2.32.

Lemma 4.5 is not entirely subsumed by Lemma 2.32. For example, define the functions f and g on $(0,\infty)$ by

$$f(x) = \frac{1}{x} \text{ and } g(x) = \frac{\sin x}{2x},$$

so

$$\sup_{x\in(0,\infty)} \frac{|g(x)|}{f(x)} = \frac{1}{2}.$$

Lemma 4.5 implies $f+g$ and $f-g$ have polynomial growth. Since $f+g$ and $f-g$ are not $\Theta(1)$ on $(0,1)$, Lemma 2.32 is not applicable to the polynomial growth of $f+g$ or $f-g$.

Composition of functions also preserves polynomial growth:

Lemma 4.6. Let f and g be polynomial-growth functions with domains A and B, respectively. If $f(A) \subseteq B$, then the function $h: A \to \boldsymbol{R}$ defined by $h(x) = g(f(x))$ also has polynomial growth.

Proof. By Lemma 2.3, we may assume h is not identically zero, so g is also not identically zero. In particular, A and B are non-empty. Lemma 2.7 implies g is positive. Lemma 2.2(1) implies A and B are positive sets. The function f is positive because $f(A) \subseteq B \subseteq \boldsymbol{R}^+$.

By Corollary 2.30, f and g can be extended to polynomial growth functions F and G, respectively, on $\boldsymbol{R}^+$. Since neither f nor g is identically zero, we conclude that neither F

nor G is identically zero. Lemma 2.7 implies F and G are positive. Define $H: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ by $H(x) = G(F(x))$.

Let $b > 1$, so $\psi_b(F) < \infty$ by Lemma 2.16. Let c be a real number such that $c \geq \psi_b(F)$ and $c > 1$. Lemma 2.16 implies $\psi_c(G) < \infty$.

Let x be any positive real number, and let L and U be the greatest lower and least upper bounds respectively for F on the interval $[x, bx]$. Corollary 2.21 implies L and U are positive real numbers. Thus $[L, U] \subseteq \boldsymbol{R}^+$.

By Lemmas 2.8,

$$\frac{U}{L} = \Lambda_F([x, bx]) \leq \psi_b(F) \leq c,$$

which implies

$$F([x, bx]) \subseteq [L, U] \subseteq [L, cL]$$

and

$$H([x, bx]) \subseteq G([L, cL]).$$

Lemmas 2.9(5) implies

$$\Lambda_H([x, bx]) \leq \Lambda_G([L, cL]) \leq \psi_c(G).$$

Therefore,

$$\psi_b(H) \leq \psi_c(G) < \infty.$$

Lemma 2.16 implies H has polynomial growth. The function h is the restriction of H to A, so Lemma 2.2(2) implies h also has polynomial growth. □

Powers of Logarithms. Let b and α be real numbers with $b > 1$. Define the function $f: (1, \infty) \to \boldsymbol{R}$ by $f(x) = \log_b^\alpha x$. If $\alpha \neq 0$, Lemmas 4.1, 4.6, and 2.19 imply f has polynomial growth on a subset D of $(1, \infty)$ if and only if $\inf D > 1$. If $\alpha = 0$, then f is the constant function $x \mapsto 1$ and has polynomial growth by Lemma 2.3.

Our results also enable us to recognize polynomial growth of powers of logarithms perturbed by the $floor$ and $ceiling$ functions. For example, the functions $g: [2, \infty) \to \boldsymbol{R}$ and $h: (\lceil b \rceil - 1, \infty) \to \boldsymbol{R}$ defined by

$$g(x) = \lceil \log_b \lfloor x \rfloor \rceil^\alpha$$

and

$$h(x) = \lfloor \log_b \lceil x \rceil \rfloor^\alpha$$

have polynomial growth.

Composition of logarithms. Lemmas 4.1(3) and 4.6 combine with Lemmas 2.3, 2.7, and 2.19 to completely determine the positive sets on which compositions of logarithms have polynomial growth.

For example, suppose D is a positive set, and let $a > 1$ and $b > 1$. Lemmas 4.1(3) and 4.6 implies $\log_b \log_a x$ has polynomial growth on D if $\inf D > a$. Lemma 2.3 implies $\log_b \log_a x$ has polynomial growth on D if $D = \{a\}$.

Conversely, suppose $\log_b \log_a x$ has polynomial growth on D. Lemma 2.7 implies $\log_b \log_a x$ must be either positive (real-valued) or identically zero on D. Either $D \subseteq (a, \infty)$ or $D = \{a\}$. If $D \subseteq (a, \infty)$, then Lemma 2.19 implies $\inf D > a$.

Similarly,

$$\log_c \log_b \log_a x$$

has polynomial growth on D for $c > 0$ if and only if either $\inf D > a^b$ or $D = \{a^b\}$.

We now determine which polynomial functions in one variable have polynomial growth on which positive sets. The result is applicable to a wider class of functions, such as

$$x^{\pi} - 2x^{\sqrt{5}} + 7x^{-1},$$

that resemble polynomials but may have negative or non-integer exponents.

Topological closure notation. $\bar{A}$ denotes the topological closure of A relative to $\boldsymbol{R}$ for a set A of real numbers.

Lemma 4.7. Define $p: \boldsymbol{R}^+ \to \boldsymbol{R}$ by

$$p(x) = \sum_{i=1}^{k} c_i x^{\alpha_i}$$

where k is a positive integer, and $c_1, \dots, c_k, \alpha_1, \dots, \alpha_k$ are real numbers. The function p has polynomial growth on a positive set D if and only if p is either positive on $\bar{D} \cap \boldsymbol{R}^+$ or identically zero on D.

Proof. Let $E = \bar{D} \cap \boldsymbol{R}^+$. Since $D \subseteq \boldsymbol{R}^+$, we have

$$\bar{D} \subseteq \overline{\boldsymbol{R}^+} = [0, \infty) = \boldsymbol{R}^+ \cup \{0\},$$

so

$$\bar{D} = E \cup (\bar{D} \cap \{0\}).$$

Suppose p has polynomial growth on D. Lemma 2.7 implies p is either positive on D or identically zero on D. Suppose p is positive on D. Lemma 2.19 combines with continuity of p to imply that $p(u) > 0$ for each positive limit point u of D. Each element of $E - D$ is a positive limit point of D, so p is positive on $E - D$. Therefore, p is positive on E as claimed.

We now prove the converse. If p is identically zero on D, then p has polynomial growth on D by Lemma 2.3. Now suppose instead that p is positive on E. Without loss of generality, we may assume $c_1, \dots, c_k$ are non-zero, and $\alpha_1, \dots, \alpha_k$ are distinct and in increasing order (combine terms with the same exponent, discard terms that are zero, and put the terms in increasing order of exponent.) Define the function g on $[0, \infty)$ by

$$g(x) = c_1 + \sum_{i=2}^{k} c_i x^{\beta_i}$$

where

$$\beta_i = \alpha_i - \alpha_1 > 0$$

for $2 \leq i \leq k$. ($0^{\beta_i} = 0$ is defined because $\beta_i > 0$.) If $k = 1$, the expression for $g(x)$ is of course interpreted as $g(x) = c_1$. We have

$$p(x) = x^{\alpha_1} g(x)$$

for all $x \in \boldsymbol{R}^+$. Corollary 2.12 implies x^{α_1} has polynomial growth on D, so Lemma 4.3 implies p has polynomial growth on D if g has polynomial growth on D. By Lemma 2.3, we may assume $k > 1$.

We now show that g is positive on $\overline{D}$: Positivity of p and x^{α_1} on E implies g is positive on E, so we may assume $\overline{D} \neq E$. Then $\overline{D} = E \cup \{0\}$, and 0 is a limit point of D. The function g is positive on D because $D \subseteq E$. Continuity of g implies $g(0) \geq 0$. Since $g(0) = c_1 \neq 0$, we conclude that $g(0) > 0$, so g is positive on $\overline{D}$.

If S is any bounded subset of D, then $\bar{S}$ is a compact subset of $\overline{D}$, so continuity of g implies the restriction of g to $\bar{S}$ has a minimum μ and a maximum M. The function g is positive on $\bar{S}$ because g is positive on $\overline{D}$, which contains $\bar{S}$. Therefore, μ is positive. The quantity M is finite because $M \in g(\bar{S})$. The set S is contained in $\bar{S}$, so

$$0 < \mu \leq g(t) \leq M < \infty$$

for all $t \in S$. In particular, g is $\Theta(1)$ on S. We conclude that g is locally $\Theta(1)$.

If D is bounded, then g is $\Theta(1)$ on D, and Corollary 2.13 implies g has polynomial growth on D. If D is unbounded, then $g|_D = \Theta(x^{\beta_k})$, which combines with Corollary 2.12 and Lemma 2.32 to imply g has polynomial growth on D. □

Examples. Define $f(x) = (x - 1)(x^2 - 3x + 3)$ on $\boldsymbol{R}$, so that $f(1) = 0$ and f is positive on $(1, \infty)$. Lemmas 4.7 and 2.2(1) imply p has polynomial growth on the closed interval $[c, \infty)$ if and only if $c > 1$. The function f does not have polynomial growth on $(1, \infty)$.

Define $g(x) = x^3 - x^2 + x$ on $\boldsymbol{R}$. The function g is positive on $\boldsymbol{R}^+$, so Lemma 4.7 implies g has polynomial growth on $\boldsymbol{R}^+$. Observe that $g(0) = 0$.

Define $h(x) = (x - \pi)^2$ on $\boldsymbol{R}$. Lemma 4.7 implies g has polynomial growth on $\boldsymbol{Z}^+$ but does not have polynomial growth on $\boldsymbol{R}^+$.

Summary. We can now instantly recognize a large class of functions as having polynomial growth. Lemmas 4.1 and 4.7 identify some important examples. Corollaries 4.3–4.4 and Lemma 4.6 provide several ways of combining polynomial-growth functions that preserve polynomial growth. Like Lemma 2.32 and Corollary 2.33, Lemma 4.5 shows that sufficiently constrained deviations from a polynomial-growth function also have polynomial growth.

Example. Let $p(x, y, z)$ and $q(x, y, z) \neq 0$ be polynomials in three variables with non-negative real coefficients. If $c > e$, the function

$$g(x) = \sqrt[3]{\frac{p(\lfloor x \rfloor, \log x, \log\log x)}{q(\lceil x \rceil, \log x, \log\log x)}}$$

on the interval $[c, \infty)$ has polynomial growth.

5. Polynomial-Growth Interpolation

In this section, we consider polynomial-growth functions on certain discrete domains and their polynomial-growth extensions to intervals. Such extensions are useful in applications of the Akra-Bazzi formula to recurrences defined on sets of integers. The main result is Corollary 5.3.

Lemma 2.16 says polynomial growth of a positive function g on a positive interval is equivalent to finiteness of $\Psi_b(g)$ for *all* $b > 1$, which is equivalent to finiteness of $\Psi_b(g)$ for *some* $b > 1$. The obvious generalization of Lemma 2.16 to positive functions on arbitrary positive sets is false, although Corollary 2.18 says $\Psi_b(g) < \infty$ for each positive polynomial-growth function g and all $b > 1$. Corollary 2.18 was followed by two examples demonstrating limitations of that proposition: A positive function on a positive set was exhibited that has finite Ψ_2 but has infinite Ψ_3 and therefore does not have polynomial growth. Another positive function on a positive set was shown to have finite Ψ_b for all $b > 1$, although the function does not have polynomial growth.

The following proposition is an analogue of Lemma 2.16 for functions on suitable discrete domains. We also provide some information about polynomial growth extensions of such functions to the minimum intervals containing their domains.

Lemma 5.1. Suppose $f: x(\boldsymbol{Z}^+) \to \boldsymbol{R}$ is a real-valued function where $x: \boldsymbol{Z}^+ \to \boldsymbol{R}^+$ is a positive, increasing sequence of real numbers. Define $z \in (0, \infty]$ by

$$z = \lim_{n\to\infty} x_n,$$

and let B be the set of all real numbers b that satisfy

$$b \geq \frac{x_{n+1}}{x_n}$$

for all sufficiently large n (so $b > 1$). Let G be the set of real-valued extensions of f to $[x_1, z)$ that are monotonic on $[x_n, x_{n+1}]$ for all positive integers n. Then G has a continuous element, and if B is non-empty (e.g., if $z < \infty$), either all or none of the following statements are true:

(1) f has polynomial growth.
(2) Either f is identically zero, or f is positive and $\Psi_b(f) < \infty$ for *some* $b \in B$.
(3) Either f is identically zero, or f is positive and $\Psi_b(f) < \infty$ for *all* $b \in B$.
(4) *Some* element of G has polynomial growth.
(5) *All* elements of G have polynomial growth.

Proof. Let $D = x(\boldsymbol{Z}^+)$, i.e., $D = domain(f)$. Since x is an increasing function, we have

$$[x_1, z) = D \cup \bigcup_{n=1}^{\infty} (x_n, x_{n+1}),$$

$$D \cap \bigcup_{n=1}^{\infty} (x_n, x_{n+1}) = \emptyset,$$

and

$$(x_i, x_{i+1}) \cap (x_j, x_{j+1}) = \emptyset$$

whenever i and j are distinct positive integers. Therefore, there exists a function $f^*: [x_1, z) \to \boldsymbol{R}$ with $f^*|_D = f$ and

$$f^*(t) = f(x_n) + \left(\frac{t - x_n}{x_{n+1} - x_n}\right)\left(f(x_{n+1}) - f(x_n)\right)$$

for all positive integers n and all $t \in (x_n, x_{n+1})$. The function f^* is continuous.

If $f(x_n) = f(x_{n+1})$, then f^* is constant on $[x_n, x_{n+1}]$. If $f(x_n) < f(x_{n+1})$, then f^* is strictly increasing on $[x_n, x_{n+1}]$. If $f(x_n) > f(x_{n+1})$, then f^* is strictly decreasing on $[x_n, x_{n+1}]$. Therefore, f^* is an element of G. We have confirmed the claim that G has a continuous element. In particular, G is non-empty, so condition (5) implies condition (4), which implies condition (1) by Lemma 2.2(2). By Lemma 2.7 and Corollary 2.18, condition (1) implies condition (3).

Now suppose B is non-empty, so condition (3) implies condition (2). We will show that condition (2) implies condition (5), and the lemma will be proved.

Assume (2) is satisfied, and let g be any element of G. For all positive integers n, monotonicity of g on $[x_n, x_{n+1}]$ implies the restriction of g to $[x_n, x_{n+1}]$ has minimum value

$$\min\{g(x_n), g(x_{n+1})\} = \min\{f(x_n), f(x_{n+1})\}$$

and maximum value

$$\max\{g(x_n), g(x_{n+1})\} = \max\{f(x_n), f(x_{n+1})\}.$$

We conclude from

$$[x_1, z) = \bigcup_{n=1}^{\infty} [x_n, x_{n+1}]$$

that g is positive if f is positive, and g is identically zero if f is identically zero. Lemma 2.3 implies g has polynomial growth if g is identically zero. Therefore, we may assume f and g are positive, and $\Psi_b(f) < \infty$ for some $b \in B$.

There exists a positive integer l with

$$\frac{x_{n+1}}{x_n} \leq b$$

for all $n \geq l$. If $z = \infty$, define $k = l$. If $z < \infty$, there exists a positive integer m with $x_n \geq z/b$ for all $n \geq m$, and we define $k = \max(l, m)$.

If $k = 1$, then $[x_1, x_k] = \{x_1\}$. If $k \neq 1$, then

$$[x_1, x_k] = \bigcup_{n=1}^{k-1} [x_n, x_{n+1}].$$

Both cases satisfy

$$\min g([x_1, x_k]) = \min_{1 \leq n \leq k} f(x_n) > 0$$

and

$$\max g([x_1, x_k]) = \max_{1 \leq n \leq k} f(x_n) < \infty,$$

so g is $\Theta(1)$ on $[x_1, x_k]$. Observe that $[x_1, x_k]$ is a lower subset of $[x_1, z)$, which is the union of $[x_1, x_k]$ and $[x_k, z)$. Furthermore $[x_1, x_k]$ has positive minimum, x_1, and finite maximum, x_k. We shall prove that g has polynomial growth on $[x_k, z)$, so g is a polynomial growth function by Corollary 2.25.

Suppose $z < \infty$, so $bx_k \geq z$ and

$$D \cap [x_k, bx_k] = \{x_n : n \geq k\}.$$

Observe that

$$[x_k, z) = \bigcup_{n=k}^{\infty} [x_n, x_{n+1}].$$

Lemma 2.10(4) implies

$$\inf\Big(g\big([x_k, z)\big)\Big) = \inf_{n \geq k}\Big(\min\big(g([x_n, x_{n+1}])\big)\Big) = \inf_{n \geq k} f(x_n) \geq \frac{f(x_k)}{\Psi_b(f)} > 0$$

and

$$\sup\Big(g\big([x_k, z)\big)\Big) = \sup_{n \geq k}\Big(\max\big(g([x_n, x_{n+1}])\big)\Big) = \sup_{n \geq k} f(x_n) \leq \Psi_b(f) f(x_k) < \infty.$$

Corollary 2.13 implies g has polynomial growth on $[x_k, z)$. Therefore, we may assume $z = \infty$.

Let y be any element of $[x_k, \infty)$, so by is also an element of $[x_k, \infty)$. There exist positive integers $v, w \geq k$ such that

$$x_v \leq y < x_{v+1} \text{ and } x_w \leq by < x_{w+1}.$$

The inequalities

$$x_{v+1} \leq bx_v \leq by < x_{w+1}$$

imply $v < w$. Define

$$H = D \cap [x_v, bx_v],$$

$$I = D \cap [x_{v+1}, bx_{v+1}],$$

and

$$J = D \cap [x_w, bx_w].$$

The sets H and I contain x_{v+1}, so $H \cap I$ is non-empty. The inequalities,

$$x_v < x_{v+1} \leq bx_v < bx_{v+1}$$

imply

$$H \cup I = D \cap [x_v, bx_{v+1}].$$

The inequalities

$$x_{v+1} \leq x_w \leq by < bx_{v+1}$$

imply I contains x_w. The set J also contains x_w, so

$$x_w \in I \cap J \subseteq (H \cup I) \cap J.$$

In particular, $(H \cup I) \cap J$ is non-empty. The inequalities

$$x_v < x_w < bx_{v+1} \leq bx_w$$

imply

$$H \cup I \cup J = D \cap [x_v, bx_w].$$

Let

$$S = D \cap [x_v, x_{w+1}] = \{x_n : \ v \leq n \leq w+1\}.$$

Since $x_{w+1} \leq bx_w$, we have $S \subseteq H \cup I \cup J$ and $f(S) \subseteq f(H \cup I \cup J)$. It follows from $H \cap I \neq \phi$ and $(H \cup I) \cap J \neq \phi$ that $f(H \cap I) \neq \phi$ and $f(H \cup I) \cap f(J) \neq \phi$. Since f is positive, the dynamic range $\Lambda_f(S)$ is defined. Parts (5) and (6) of Lemma 2.9 imply

$$\Lambda_f(S) \leq \Lambda_f(H \cup I \cup J) \leq \Lambda_f(H \cup I)\Lambda_f(J) \leq \Lambda_f(H)\Lambda_f(I)\Lambda_f(J) \leq \Psi_b(f)^3.$$

Define $L = \inf g([x_v, x_{w+1}])$ and $U = \sup g([x_v, x_{w+1}])$, so

$$L = \min_{v \leq n \leq w} \big(\min\big(g([x_n, x_{n+1}])\big)\big) = \min f(S)$$

and

$$U = \max_{v \le n \le w} \left(\max\left(g([x_n, x_{n+1}]) \right) \right) = \max f(S).$$

Lemmas 2.8 and 2.9(5) combine with $[y, by] \subseteq [x_v, x_{w+1}]$ to imply

$$\Lambda_g([y, by]) \le \Lambda_g([x_v, x_{w+1}]) = \frac{U}{L} = \frac{\max f(S)}{\min f(S)} = \Lambda_f(S) \le \Psi_b(f)^3.$$

Therefore,

$$\Psi_b\left(g|_{[x_k, \infty)}\right) = \sup_{y \in [x_k, \infty)} \Lambda_g([y, by]) \le \Psi_b(f)^3 < \infty.$$

Lemma 2.16 implies g has polynomial growth on $[x_k, \infty)$ as required. □

Limit superior of ratios. Let B and $x_1, x_2, x_3, \ldots$ be as in Lemma 5.1, and define

$$L = \limsup_{n \to \infty} \frac{x_{n+1}}{x_n}.$$

Either $B = (L, \infty)$ or $B = [L, \infty)$. In particular, B is non-empty if and only if $L < \infty$.

Corollary 5.2. Let $f: D \to \boldsymbol{R}$ be a real-valued function on a non-empty upper subset D of the positive integers. Let G be the set of real-valued extensions of f to $[\min D, \infty)$ that are monotonic on $[n, n+1]$ for all $n \in D$. Then G has a continuous element, and either all or none of the following statements are true:

(1) f has polynomial growth.
(2) Either f is identically zero, or f is positive and $\Psi_b(f) < \infty$ for *some* $b > 1$.
(3) Either f is identically zero, or f is positive and $\Psi_b(f) < \infty$ for *all* $b > 1$.
(4) *Some* element of G has polynomial growth.
(5) *All* elements of G have polynomial growth.

Proof. Define the surjection $x: \boldsymbol{Z}^+ \to D$ by $x_n = n - 1 + \min D$. Since x is an increasing function and

$$\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = \lim_{n \to \infty} \frac{n + \min D}{n - 1 + \min D} = 1,$$

the interval $(1, \infty)$ is the set of all real numbers b that satisfy

$$\frac{x_{n+1}}{x_n} \le b$$

for all sufficiently large n. The proposition follows from Lemma 5.1. □

Corollary 5.3. If f is a polynomial-growth function on a set of integers, then f has a continuous, polynomial-growth extension to $\boldsymbol{R}^+$.

Proof. Let $D = domain(f)$. Lemma 2.2(1) implies is f is real-valued and D is a positive set. If f is identically zero, then the identically zero function on $\boldsymbol{R}^+$ is a continuous extension of f and has polynomial growth by Lemma 2.3. Now suppose f is not identically zero, so D is non-empty. Lemma 2.7 implies f is positive.

Lemmas 2.30 and 2.2(1) implies f can be extended to a polynomial-growth function $f^*: \boldsymbol{Z}^+ \to \boldsymbol{R}$. Corollary 5.2 implies f^* can be extended to a continuous, polynomial-growth function $g: [1, \infty) \to \boldsymbol{R}$, which is also an extension of f. Positivity of f and non-emptiness of D imply g is not identically zero. Lemma 2.7 implies g is positive.

Define $g^*: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ by $g^*|_{[1,\infty)} = g$ and $g^*(x) = g(1)$ for all $x \in (0,1)$. The function g^* is continuous. Lemma 2.3 implies g^* has polynomial growth on $(0,1)$. Lemma 2.24 implies g^* is a polynomial-growth function. Furthermore, g^* is an extension of f because g^* is an extension of g, which is an extension of f. □

Infinitely Differentiable Extensions. Partly because of Leighton's remark in [Le] about derivatives and polynomial growth, and partly just for fun, we will show in Corollary 5.6 that the set G of Lemma 5.1 has an infinitely differentiable element g. Corollaries 5.7 and 5.8 replace continuity with infinite differentiability for Corollaries 5.2 and 5.3, respectively.

The bridging function J of Lemma 5.4 below is the key building block for g. We use the same well-known construction for J as in [Wik]. An alternative choice for J is provided by problem 12 on page 40 of [GO].

Lemma 5.4. There exists an infinitely differentiable function $J: \boldsymbol{R} \to [0,1]$ such that

(1) $J(x) = 0$ for all $x \leq 0$.

(2) $J(x) = 1$ for all $x \geq 1$.

(3) $J|_{[0,1]}$ is strictly increasing.

Proof. Define a function $f: \boldsymbol{R} \to \boldsymbol{R}$ by

$$f(x) = \begin{cases} e^{-1/x}, & \text{for } x > 0 \\ 0, & \text{for } x \leq 0. \end{cases}$$

The function f is infinitely differentiable on $(-\infty, 0)$ with $f^{(n)}(x) = 0$ for all $x < 0$ and all $n \in \boldsymbol{Z}^+$. Therefore the left nth derivative of f is defined at zero with value zero.

The function f is also infinitely differentiable on $(0, \infty)$: It can be easily shown by induction that for each nonnegative integer n, the function f is n times differentiable on $(0, \infty)$ and there exists a polynomial p_n with real coefficients such that the nth derivative $f^{(n)}$ satisfies

$$f^{(n)}(x) = \frac{p_n(x)}{x^{2n}}\, e^{-1/x}$$

for all $x > 0$. Here $f^{(0)} = f$ and $p_0(x) = 1$.

We claim that f is infinitely differentiable at zero, and $f^{(n)}(0) = 0$ for each non-negative integer n. The case $n = 0$ is a restatement of the definition of f at zero, i.e., $f(0) = 0$. If $n \geq 0$ is an integer for which f is n time differentiable at zero and $f^{(n)}(0) = 0$, then the $(n + 1)$th right derivative is defined at 0 by

$$\lim_{x \to 0^+} \frac{f^{(n)}(x)}{x} = \lim_{x \to 0^+} \frac{p_n(x)}{x^{2n+1}}\, e^{-1/x} = 0,$$

which agrees with the $(n + 1)$th left derivative at zero. Thus f is $n + 1$ times differentiable at zero and $f^{(n+1)}(0) = 0$. The claim follows by induction. Therefore, f is infinitely differentiable on $\boldsymbol{R}$.

Since

$$f(x) + f(1 - x) > 0$$

for all $x \in \boldsymbol{R}$, we may define a function $J: \boldsymbol{R} \to \boldsymbol{R}$ by

$$J(x) = \frac{f(x)}{f(x) + f(1 - x)}.$$

The function J is infinitely differentiable because f is infinitely differentiable. If $x \leq 0$, then $J(x) = 0$ because $f(x) = 0$. If $x \geq 1$, then $J(x) = 1$ because $f(1 - x) = 0$. The derivative of J satisfies

$$J'(x) = \frac{f'(x)\big(f(x) + f(1 - x)\big) - f(x)\big(f'(x) - f'(1 - x)\big)}{\big(f(x) + f(1 - x)\big)^2}$$

$$= \frac{f'(x)f(1 - x) + f(x)f'(1 - x)}{\big(f(x) + f(1 - x)\big)^2}.$$

If $x \in (0,1)$, then $1 - x \in (0,1)$. Since f and f' are positive on $(0,1)$, we conclude from the expression above for J' that J' is positive on $(0,1)$. Therefore $J|_{[0,1]}$ is strictly increasing. Continuity of J implies $J([0,1]) = [0,1]$. Therefore, $J(\boldsymbol{R}) = [0,1]$. □

Definition. A real-valued function on a set of real numbers is strictly monotonic if it is either strictly increasing or strictly decreasing.

Corollary 5.5. Let a, b, c, and d be real numbers such that $a < b$. There exists an infinitely differentiable function $h: [a, b] \to \boldsymbol{R}$ such that

(1) $h(a) = c$.

(2) $h(b) = d$.

(3) h is either constant or strictly monotonic.

(4) For each positive integer n, the nth derivative satisfies

$$h^{(n)}(a) = h^{(n)}(b) = 0.$$

Proof. Let J be as in Lemma 5.4, and define the functions $\lambda: [a, b] \to [0,1]$ and $h: [a, b] \to \boldsymbol{R}$ by

$$\lambda(x) = \frac{x - a}{b - a}$$

and

$$h(x) = c + J\big(\lambda(x)\big)(d - c).$$

Observe that $h(a) = c$ and $h(b) = d$, as required. If $c = d$, then $h(x) = c$ for all $x \in [a, b]$. Since λ and $J|_{[0,1]}$ are strictly increasing, h is strictly increasing if $c < d$, and h is strictly decreasing if $c > d$.

The function h is infinitely differentiable. For all $x \in [a, b]$ and $n \in \boldsymbol{Z}^+$, the nth derivative is given by

$$h^{(n)}(x) = \frac{J^{(n)}\big(\lambda(x)\big)(d - c)}{(b - a)^n}.$$

Therefore,

$$h^{(n)}(a) = \frac{J^{(n)}(0)(d - c)}{(b - a)^n} = 0$$

and

$$h^{(n)}(b) = \frac{J^{(n)}(1)(d - c)}{(b - a)^n} = 0.$$

□

Corollary 5.6. Let $x: \boldsymbol{Z}^+ \to \boldsymbol{R}^+$ be a positive, increasing sequence of real numbers, and define $z \in (0, \infty]$ by

$$z = \lim_{n \to \infty} x_n.$$

Given a function $f: x(\boldsymbol{Z}^+) \to \boldsymbol{R}$, there exists an infinitely differentiable function $g: [x_1, z) \to \boldsymbol{R}$ such that for all $n \in \boldsymbol{Z}^+$,

(1) $g(x_n) = f(x_n)$.

(2) The restriction of g to $[x_n, x_{n+1}]$ is either constant or strictly monotonic.

(3) $g^{(k)}(x_n) = 0$ for all $k \in \boldsymbol{Z}^+$.

Proof. Corollary 5.5 implies that for all $n \in \boldsymbol{Z}^+$ there exists an infinitely differentiable function $h_n: [x_n, x_{n+1}] \to \boldsymbol{R}^+$ such that

(a) $h_n(x_n) = f(x_n)$ and $h_n(x_{n+1}) = f(x_{n+1})$.

(b) h_n is either constant or strictly monotonic.

(c) $h_n^{(k)}(x_n) = h_n^{(k)}(x_{n+1}) = 0$ for all $k \in \boldsymbol{Z}^+$.

Since

$$[x_1, z) = \dot{\bigcup_{n \in \boldsymbol{Z}^+}} [x_n, x_{n+1})$$

is a union of disjoint sets, there exists a function $g: [x_1, z) \to \boldsymbol{R}$ such that

$$g|_{[x_n, x_{n+1})} = h_n|_{[x_n, x_{n+1})}$$

for all $n \in \boldsymbol{Z}^+$. (We are using the axiom of choice here, although its use can be avoided.) In particular,

$$g(x_n) = h_n(x_n) = f(x_n).$$

Since each h_n is infinitely differentiable, g is infinitely differentiable on each interval of the form (x_n, x_{n+1}). Let $k \in \boldsymbol{Z}^+$. The kth right derivative of g at x_n is inherited from h_n with the value zero. For $n > 1$, the kth left derivative of g at x_n is inherited from h_{n-1} and is also zero. Therefore the kth left and right derivatives of g at x_n are defined and in agreement when $n > 1$. Of course, the kth derivative at x_1 is one sided. We conclude that $g^{(k)}(x_n) = 0$ for all $n \in \boldsymbol{Z}^+$. The function g is infinitely differentiable.

By construction, g agrees with h_n on each $[x_n, x_{n+1})$. It follows from

$$g(x_{n+1}) = f(x_{n+1}) = h_n(x_{n+1})$$

that g agrees with h_n on $[x_n, x_{n+1}]$. Therefore, the restriction of g to $[x_n, x_{n+1}]$ is either constant or strictly monotonic. □

Corollary 5.7. Let $f: D \to \boldsymbol{R}$ be a real-valued function on a non-empty upper subset D of the positive integers. There exists an infinitely differentiable extension g of f to $[\min D, \infty)$ such that for all $n \in D$, g is monotonic on $[n, n+1]$ and $g^{(k)}(n) = 0$ for each positive integer k where $g^{(k)}$ is the kth derivative of g.

Proof. The proposition follows from Corollary 5.6 with $x: \boldsymbol{Z}^+ \to D$ defined by

$$x_n = n - 1 + \min D,$$

so that x_n approaches infinity as n approaches infinity. □

Example. Define the function $p: [1, \infty) \to \boldsymbol{R}$ by

$$p(x) = (2x - 3)(4x - 7).$$

The roots of p are $3/2$ and $7/4$. The function p is positive on $(7/4, \infty)$. Let f be the restriction of p to $\boldsymbol{Z}^+$. The inequality $f(1) > 0$ implies f is a positive function. Lemma 4.7 implies f has polynomial growth but p does not. Corollary 5.7 implies there exists an infinitely differentiable extension g of f to $[1, \infty)$ that is monotonic on $[n, n+1]$ for all $n \in \boldsymbol{Z}^+$ and has vanishing derivatives of all positive orders at each such n. Corollary 5.2 implies g has polynomial growth.

The function p is not monotonic on the interval $[1,2]$, which contains both roots. Furthermore, neither its derivative $x \mapsto 16x - 26$ nor its second derivative $x \mapsto 16$ vanish at any positive integers. The function g is not the obvious infinitely differentiable extension p of f.

Corollary 5.8. If f is a polynomial-growth function on a set of integers, then f has an infinitely differentiable, polynomial-growth extension to $\boldsymbol{R}^+$.

Proof. Our argument is an obvious adaptation of the proof of Corollary 5.3. Let D be the domain of f. Lemma 2.2(1) implies is f is real-valued and D is a positive set. If f is identically zero, then the identically zero function on $\boldsymbol{R}^+$ is an infinitely differentiable extension of f and has polynomial growth by Lemma 2.3. Now suppose f is not identically zero, so D is non-empty. Lemma 2.7 implies f is positive.

Lemma 2.30 implies there exists a polynomial-growth function f^* on $\boldsymbol{Z}^+$ that is an extension of f. Lemma 2.2(1) implies f^* is real-valued.

Corollary 5.7 implies there exists an infinitely differentiable extension g of f^* (and f) to $[1, \infty)$ such that for all $n \in \boldsymbol{Z}^+$, g is monotonic on $[n, n+1]$ and $g^{(k)}(n) = 0$ for each positive integer k where $g^{(k)}$ is the kth derivative of g. Lemma 5.2 implies g has polynomial growth. Positivity of f and non-emptiness of D imply g is not identically zero. Lemma 2.7 implies g is positive.

Define $h: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ by $h|_{[1,\infty)} = g$ and $h(x) = g(1)$ for all $x \in (0,1)$. Lemma 2.3 implies h has polynomial growth on $(0,1)$. Lemma 2.24 implies h is a polynomial-growth function. The function h is an extension of f because h is an extension of g, which is an extension of f. The restriction of h to $(0, 1]$ is constant and is therefore infinitely differentiable. For each positive integer k, the kth left derivative of h at 1 is 0, which agrees the kth derivative of g at 1, i.e., the kth right derivative of h at 1. Therefore, h is infinitely differentiable at 1. We conclude that h is an infinitely differentiable function. □

6. Leighton's Second Example

The second example in [Le] of Theorem 1 says

"If $T(x) = 2T(x/2) + \frac{8}{9}T(3x/4) + \Theta(x^2/\log x)$, then $p = 2$ and $\mathrm{T}(x) = \Theta(x^2/\log\log x)$."

The conclusion is incorrect, as we shall see. The recurrence in Leighton's example is presumably shorthand for the family of recurrences of the form

$$T(x) = \begin{cases} \Theta(1), & \text{for } 1 \le x \le x_0 \\ 2T(x/2) + \dfrac{8}{9}T(3x/4) + g(x), & \text{for } x > x_0. \end{cases}$$

that satisfy

$$g(x) = \Theta(x^2/\log x)$$

and the hypothesis of Theorem 1, i.e.,

$$x_0 \ge \max\left\{\frac{1}{1/2}, \frac{1}{3/4}, \frac{1}{1/4}\right\} = 4$$

and g is a non-negative (and locally Riemann integrable) real-valued function satisfying Leighton's polynomial-growth condition relative to $\{1/2\,, 3/4\}$.

His polynomial-growth condition requires the domain of g to contain the interval $[1/2,\infty)$. The extraneous inclusion of $[1/2, x_0]$ in the domain of g means that we cannot replace $\Theta(x^2/\log x)$ with $x^2/\log x$ for any member of the family. Non-negativity of g implies $g(x) \ne x^2/\log x$ for all $x \in [1/2, 1)$. Furthermore, $x^2/\log x$ does not represent a real number when $x = 1$.

Let $x_0 \ge 4$. For each real-valued function f on $[1/2,\infty)$, define the real-valued function T_f on $[1, \infty)$ by

$$T_f(x) = \begin{cases} 1, & \text{for } 1 \le x \le x_0 \\ 2T_f(x/2) + \frac{8}{9}T_f(3x/4) + f(x), & \text{for } x > x_0. \end{cases}$$

Let h be the continuous real-valued function on $[1/2,\infty)$ defined by

$$h(x) = \begin{cases} x^2/\log x\,, & \text{for } x \ge x_0 \\ {x_0}^2/\log x_0\,, & \text{for } 1/2 \le x < x_0. \end{cases}$$

The defining recurrence for T_h is a member of the family of recurrences referenced by Leighton's example. Theorem 1 of [Le] implies

$$T_h(x) = \Theta\left(x^2\left(1 + \int_1^x \frac{h(u)}{u^3}\,du\right)\right) = \Theta\left(x^2 \int_{x_0}^x \frac{1}{u\log u}\,du\right),$$

i.e.,

$$T_h(x) = \Theta(x^2 \log\log x) \ne \Theta(x^2/\log\log x),$$

which contradicts [Le].

Theorem 1 of [Le] has excess baggage that unnecessarily complicates its application to this family of recurrences. For example, let α be any non-negative real-valued function on $[1/2,\infty)$ such that $\alpha(x) = x^2/\log x$ for each $x > 1$, so that α approaches ∞ as x decreases to 1. Lemma 2.19 implies α does not have polynomial growth regardless of how we define the restriction of α to $[1/2\,,1]$. By Corollary 2.17, α does not satisfy Leighton's polynomial-growth condition relative to $\{1/2\,,3/4\}$. Therefore, Theorem 1 is not *directly* applicable to our *description* of the recurrence for T_α. Furthermore, unboundedness of the Akra-Bazzi integrand

$$\frac{\alpha(u)}{u^3}$$

on $[1,x]$ for all $x > 1$ implies the integrand is not Riemann integrable on any such interval. To make matters worse, the inapplicable Akra-Bazzi integral

$$\int_1^x \frac{\alpha(u)}{u^3}\,du$$

is divergent as an improper integral:

$$\lim_{t\to1^+}\int_t^x \frac{\alpha(u)}{u^3}\,du = \lim_{t\to1^+}\int_t^x \frac{1}{u\log u}\,du = \log\log x - \lim_{t\to1^+}(\log\log t) = \infty.$$

However, $\alpha|_{(x_0,\infty)} = h|_{(x_0,\infty)}$. Therefore, the functions T_α and T_h are identical, and

$$T_\alpha(\mathrm{x}) = T_h(\mathrm{x}) = \Theta(x^2\log\log x).$$

7. Recurrences

We start with a discussion of difference equations as examples of recurrence relations, followed by definitions of multi-recurrences, divide-and-conquer recurrences, and their solutions. We shall also describe the relationship of our definitions to [Le].

Some awareness of linear algebra is assumed. Herstein's classic text [He] is an excellent resource.

Shift operators. Let F be a field, and let V be the vector space over F of all infinite sequences $x: \boldsymbol{Z}^+ \to F$ with members in F. The *left shift operator* on V is the function $L: V \to V$ defined by

$$\left(L(x)\right)_n = x_{n+1}$$

for all $n \in \boldsymbol{Z}^+$, i.e., the sequence

$$x_1, x_2, x_3, \ldots$$

is mapped to the sequence

$$x_2, x_3, x_4, \ldots$$

The *right shift operator* on V is the function $R: V \to V$ defined by

$$\left(R(x)\right)_n = \begin{cases} 0, & \text{for } n = 1 \\ x_{n-1}, & \text{for } n > 1 \end{cases}$$

for all $n \in \boldsymbol{Z}^+$, i.e., the sequence

$$x_1, x_2, x_3, \ldots$$

is mapped to the sequence

$$0, x_1, x_2, x_3, \ldots$$

Both shift operators are linear transformations. Each non-zero $\lambda \in F$ is an eigenvalue of L with an associated eigenvector

$$1, \lambda, \lambda^2, \lambda^3, \ldots$$

The null space of $L - \lambda I$ is one dimensional, i.e., every λ-eigenvector of L is a scalar multiple of the eigenvector shown above. Here I is the identity transformation on V.

Fibonacci Numbers. The most famous recurrence relation is undoubtedly the definition of the nth Fibonacci number as

$$F_n = \begin{cases} 1, & \text{for } n = 1 \text{ and } n = 2 \\ F_{n-1} + F_{n-2}, & \text{for } n > 2, \end{cases}$$

which yields the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55 \ldots$$

(Many authors also include $F_0 = 0$ as a Fibonacci number.) We shall review a very natural derivation via linear algebra of the well-known formula for the nth Fibonacci number. A popular alternative method uses generating functions (see [Wilf] and [GKP]). The formula can also be easily proved by induction. However, induction does not explain how to discover the formula.

Let $F\colon \boldsymbol{Z}^+ \to \boldsymbol{R}$ be the function $n \mapsto F_n$, and let V be the vector space over $\boldsymbol{R}$ of all real-valued functions defined on the positive integers. Let L be the left shift operator on V. Let W be the null space of $L^2 - L - I$, where I is the identity map on V, i.e., W is the set of all sequences $w\colon \boldsymbol{Z}^+ \to \boldsymbol{R}$ that satisfy

$$w_{n+2} = w_{n+1} + w_n$$

for each positive integer n. In particular, $F \in W$ and W is L-invariant. The polynomial $t^2 - t - 1$ has roots

$$\varphi = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \psi = \frac{1 - \sqrt{5}}{2}.$$

The number φ is called the *golden ratio.* Let E_φ and E_ψ be the eigenvectors

$$1, \varphi, \varphi^2, \varphi^3, \ldots$$

and

$$1, \psi, \psi^2, \psi^3, \ldots$$

of L associated with the eigenvalues φ and ψ, respectively. In particular, E_φ and E_ψ are linearly independent elements of W.

Let $\boldsymbol{R}^2$ be the real vector space of ordered pairs of real numbers, and define $\pi\colon W \to \boldsymbol{R}^2$ by $\pi(w) = (w_1, w_2)$ for all $w \in W$. The function π is a linear transformation. For each $(c_1, c_2) \in \boldsymbol{R}^2$ there is exactly one element w of W that satisfies $w_1 = c_1$ and $w_2 = c_2$, i.e.,

$$w_n = \begin{cases} c_1, & \text{for } n = 1 \\ c_2, & \text{for } n = 2 \\ w_{n-1} + w_{n-2}, & \text{for } n > 2. \end{cases}$$

Therefore, π is a vector space isomorphism of W onto $\boldsymbol{R}^2$, so W has dimension 2, which implies $\{E_\varphi, E_\psi\}$ is a basis of W. We conclude that F is a linear combination of E_φ and E_ψ, i.e.,

$$F = aE_\varphi + bE_\psi$$

for some $a, b \in \boldsymbol{R}$, so

$$F_n = a\varphi^{n-1} + b\psi^{n-1}$$

for all positive integers n. In particular,

$$a + b = F_1 = 1$$

and

$$1 = F_2 = a\varphi + b\psi = a\varphi + (1 - a)\psi.$$

Then

$$a = \frac{1 - \psi}{\varphi - \psi} = \frac{\varphi}{\sqrt{5}}$$

and

$$b = 1 - a = \frac{\sqrt{5} - \varphi}{\sqrt{5}} = -\frac{\psi}{\sqrt{5}}.$$

Therefore,

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

for all positive integers n.

Much of the discussion of Fibonacci numbers has an obvious generalization (with appropriate modifications) to any homogeneous linear difference equation with constant coefficients over any field. Many such equations involve repeated eigenvalues. Further details are provided in the next few pages.

Generalized eigenvectors of left shift operators. Let F be a field, and let V be the vector space over F of all infinite sequences $x: \boldsymbol{Z}^+ \to F$ with members in F. Let L and R be the left and right shift operators, respectively, on V. Let λ be any element of F, so the null space of $L - \lambda I$ is the one-dimensional subspace of V spanned by

$$1, \lambda, \lambda^2, \lambda^3, \ldots$$

Here I is the identity map on V. We shall use R and binomial coefficients to specify generalized eigenvectors of L corresponding to the eigenvalue λ. For each non-negative integer m, define $s_m: \boldsymbol{Z}^+ \to F$ by

$$s_m(n) = \binom{m + n - 1}{m} \lambda^{n-1}$$

for all positive integers n, i.e., s_m is the sequence

$$\binom{m}{m}, \binom{m+1}{m}\lambda, \binom{m+2}{m}\lambda^2, \binom{m+3}{m}\lambda^3, \ldots$$

Also define $b_m: \boldsymbol{Z}^+ \to F$ by $b_m = R^m(s_m)$. Here R^m refers to composition of functions instead of exponentiation of function values, and $R^0 = I$. The first few b_m are shown below:

$$b_0 = \binom{0}{0}, \binom{1}{0}\lambda, \binom{2}{0}\lambda^2, \binom{3}{0}\lambda^3, \ldots = 1, \lambda, \lambda^2, \lambda^3, \ldots$$

$$b_1 = 0, \binom{1}{1}, \binom{2}{1}\lambda, \binom{3}{1}\lambda^2, \binom{4}{1}\lambda^3, \ldots = 0, 1, 2\lambda, 3\lambda^2, 4\lambda^3, \ldots$$

$$b_2 = 0, 0, \binom{2}{2}, \binom{3}{2}\lambda, \binom{4}{2}\lambda^2, \binom{5}{2}\lambda^3, \ldots = 0, 0, 1, 3\lambda, 6\lambda^2, 10\lambda^3, \ldots$$

$$b_3 = 0, 0, 0, \binom{3}{3}, \binom{4}{3}\lambda, \binom{5}{3}\lambda^2, \binom{6}{3}\lambda^3, \ldots = 0, 0, 0, 1, 4\lambda, 10\lambda^2, 20\lambda^3, \ldots$$

$$b_4 = 0, 0, 0, 0, \binom{4}{4}, \binom{5}{4}\lambda, \binom{6}{4}\lambda^2, \binom{7}{4}\lambda^3, \ldots = 0, 0, 0, 0, 1, 5\lambda, 15\lambda^2, 35\lambda^3, \ldots$$

...

Let $a = (L - \lambda I)(b_m)$ for some positive integer m, so

$$a_n = b_m(n+1) - \lambda b_m(n).$$

for each positive integer n. If $n < m$, then

$$a_n = 0 = b_{m-1}(n).$$

If $n = m$, then

$$a_n = 1 = b_{m-1}(n).$$

If $n > m$, then

$$a_n = s_m(n - m + 1) - \lambda s_m(n - m),$$

i.e.,

$$a_n = \binom{n}{m}\lambda^{n-m} - \binom{n-1}{m}\lambda^{n-m} = \binom{n-1}{m-1}\lambda^{n-m},$$

so

$$a_n = s_{m-1}(n - m + 1) = b_{m-1}(n).$$

Therefore, $a = b_{m-1}$, i.e.,

$$(L - \lambda I)(b_m) = b_{m-1}$$

for all positive integers m. Recall that $(L - \lambda I)(b_0) = 0$. For all positive integers k, we have

$$(L - \lambda I)^k(b_k) = b_0 \neq 0$$

and

$$(L-\lambda I)^{k+1}(b_k) = (L-\lambda I)\Big((L-\lambda I)^k(b_k)\Big) = (L-\lambda I)(b_0) = 0.$$

For each such k, let W_k be the null space of $(L-\lambda I)^k$, so

$$b_k \in W_{k+1} - W_k.$$

Since $b_0 \in W_1$, we conclude that

$$b_0, \dots, b_{k-1}$$

are linearly independent elements of W_k. Let α be any positive integer, $W = W_\alpha$, and $N = L - \lambda I$, so W is the null space of N^α and is therefore N-invariant. (N is a mnemonic for *nilpotent*). Define

$$B = \{b_0, \dots, b_{\alpha-1}\}.$$

We shall show that B is a basis for W, and W has dimension α. Let Y be the space spanned by B, so $Y \subseteq W$. The set B is a basis for Y, which has dimension α. Since $N(b_0) = 0 \in Y$ and $N(b_i) = b_{i-1} \in Y$ for all $b_i \in B - \{b_0\}$, we conclude that $N(B) \subseteq Y$, which implies $N(Y) \subseteq Y$.

Suppose $Y \neq W$, so there exists $z \in W - Y$. The set

$$C = \{N^k(z) : 0 \le k \le \alpha\}$$

contains z because $N^0 = I$, and C is contained in W because W is N-invariant. Furthermore, C is N-invariant because $N^{\alpha+1}(z) = 0 = N^\alpha(z) \in C$. Let U be the space spanned by Y and C. The space Y is finite dimensional, and C is finite, so U is finite dimensional. Let d be the dimension of U. The space U is N-invariant because Y and C are N-invariant. We know $U \subseteq W$ because $Y \subseteq W$ and $C \subseteq W$. The space Y is properly contained in U because $z \in C - Y \subseteq U - Y$. Therefore, $d > \alpha$.

The space U is the direct sum of N-invariant subspaces $H_1, \dots, H_q$ for some positive integer q such that the characteristic polynomial $p_i(t) \in F[t]$ of $N|_{H_i}$ is also the minimum polynomial of $N|_{H_i}$. Each H_i is annihilated by N^α because $H_i \subseteq U \subseteq W$. Therefore, the monic polynomial $p_i(t)$ divides the polynomial $t^\alpha \in F[t]$. We conclude that $p_i(t) = t^{\beta_i}$ for some positive integer $\beta_i \le \alpha$. Furthermore, β_i is the dimension of H_i, so

$$d = \sum_{i=1}^{q} \beta_i.$$

We conclude from $\beta_i \le \alpha < d$ that $q > 1$. The restriction of N to U has characteristic polynomial t^d because

$$t^d = \prod_{i=1}^{q} t^{\beta_i}.$$

The N-invariant subspaces H_1 and H_2 contain eigenspaces E_1 and E_2, respectively, with associated eigenvalue 0, i.e., E_1 and E_2 are non-zero subspaces of the null space of N. However, the null space of N is the one-dimensional space spanned by b_0, so

$$0 \neq b_0 \in E_1 \cap E_2 \subseteq H_1 \cap H_2,$$

which is a contradiction. Therefore, $Y = W$. We conclude that B is a basis for W_α, which has dimension α.

Homogeneous linear difference equations with constant coefficients. We can now generalize our discussion of Fibonacci numbers. A homogeneous linear difference equation with constant coefficients in a field F is synonymous over F with an equation of the form

$$\big(p(L_F)\big)(u) = 0$$

where L_F is the left shift operator on the vector space V_F of sequences $v\colon \boldsymbol{Z}^+ \to F$ with members in F, and $p(x) \in F[x]$, i.e., $p(x)$ is a polynomial in one indeterminate with coefficients in F. Here u is a solution in V_F of the difference equation. In other words, the null space, W_F, of the linear transformation $p(L_F)$ is the solution set in V_F of the difference equation.

Let K be a field extension of F such that $p(x)$ is a product of linear factors in $K[x]$, $i.e.$, K contains a splitting field of the polynomial $p(x)$. (Existence of such extensions is guaranteed.) Let L_K be the left shift operator on the space V_K of sequences $v\colon \boldsymbol{Z}^+ \to K$ with members in K. Let W_K be the null space of $p(L_K)$, i.e., W_K is the set of solutions in V_K of the difference equation. Then $W_F = W_K \cap V_F$.

Let d be the degree of $p(x)$. Each initial sequence $v_1, \dots, v_d$ of elements of K has exactly one extension to a solution of the difference equation in V_K, i.e., a unique extension to an element of W_K. Thus the linear transformation $\pi_K\colon W_K \to K^d$ defined by $v \mapsto (v_1, \dots, v_d)$ is an isomorphism, which implies the dimension of W_K is d. (The analogous map for W_F is also an isomorphism, so W_F also has dimension d.)

The space W_K is L_K-invariant. Let T be the restriction of L_K to W_K, so $T\colon W_K \to W_K$ is a linear transformation, and $p(T) = 0$. Let

$$p(x) = \prod_{i=1}^{s} (x - \lambda_i)^{\alpha_i}$$

be the representation of $p(x)$ as a product of powers of distinct linear factors. The space W_K is the direct sum of the subspaces $J_1, \dots, J_s$ where J_i is the null space of $(T - \lambda_i I)^{\alpha_i}$. For $1 \le i \le s$ and $0 \le m < \alpha_i$ define $s_{i,m}: \boldsymbol{Z}^+ \to K$ and $b_{i.m}: \boldsymbol{Z}^+ \to K$ by

$$s_{i,m}(n) = \binom{m+n-1}{m} \lambda_i^{n-1}$$

and

$$b_{i,m} = R_K^m(s_{i,m})$$

where R_K is the right shift operator on V_K. Let

$$B_i = \{b_{i,0}, \dots, b_{i,\alpha_i - 1}\}.$$

As explained earlier, B_i is a basis of J_i. Define

$$B = \bigcup_{i=1}^{s} B_i,$$

so B is a basis of W_K. The set $\pi_K(B)$ is a basis of $\pi_K(W) = K^d$ because π_K is an isomorphism. Each $(v_1, \dots, v_d) \in K^d$ has a unique representation as a K-linear combination of the elements of $\pi_K(B)$:

$$(v_1, \dots, v_d) = \sum_{i=1}^{s} \sum_{j=0}^{\alpha_i - 1} c_{i,j} \pi_K(b_{i,j})$$

where each $c_{i,j}$ is an element of K. The coefficients $c_{i,j}$ can be found by solving the system of linear equations above, provided each $b_{i,j}$ is known, i.e., the roots of $p(x)$ are known. Let

$$v = \sum_{i=1}^{s} \sum_{j=0}^{\alpha_i - 1} c_{i,j} b_{i,j},$$

so $v \in W_K$ and

$$\pi_K(v) = (v_1, \dots, v_d).$$

In other words, v is the unique solution in W_K of the difference equation with initial subsequence $v_1, \dots, v_d$.

Difference equations on arbitrary non-empty upper subsets of the integers. Our discussion of sequences, shift operators, and difference equations on the positive integers is applicable, *mutatis mutandis,* to domains that are non-empty upper subsets of the integers. Only a simple change of variables is required.

The difference operator. We now justify our description of solutions of homogeneous linear difference equations with constant coefficients as null spaces of linear transformations defined as polynomials applied to left shift operators. Let V be the vector space over a field F of all infinite sequences $x: \boldsymbol{Z}^+ \to F$. Let L be the left shift operator on V, and define the difference operator $\Delta: V \to V$ by

$$\Delta(x)(n) = x_{n+1} - x_n$$

for all positive integers n. Let $F[\Delta]$ and $F[L]$ be the rings of polynomials in Δ and L, respectively. We have

$$\Delta = L - I \in F[L],$$

so $F[\Delta] \subseteq F[L]$. Similarly,

$$L = \Delta + I \in F[\Delta],$$

so $F[\mathrm{L}] \subseteq F[\Delta]$. Therefore, $F[\Delta] = F[L]$. (Here $I: V \to V$ is the identity map.)

Difference equations are very different from divide-and-conquer recurrences. However, we shall discover in Section 35 that solutions of many difference equations of the form

$$\big(p(L)\big)(v) = G(v)$$

have their asymptotic behavior determined by an application of the Akra-Bazzi formula to an associated divide-and-conquer recurrence. Here $v: \boldsymbol{Z}^+ \to \boldsymbol{R}$ is an element of the vector space V of real infinite sequences, $p \in \boldsymbol{R}[x]$ is a polynomial with real coefficients, L is the left shift operator on V, and the function $G: V \to V$ maps sequences to sequences. Unlike our previous examples, the difference equation may be nonhomogeneous, i.e., G is not assumed to be identically zero. Furthermore, G need not be constant.

We define *multi-recurrences* with enough generality to include (with some adaptation) nearly all our examples that satisfy the conditions of [Le]. (Section 19 contains an example that violates part (6) of the definition because its sole dependency's range is not contained in the recurrence's domain.) We later define semi-divide-and-conquer recurrences and show they are representable as multi-recurrences. The previously discussed difference equations are also representable as such.

Definition. A *multi-recurrence* is a $(k+5)$-tuple

$$(D, C, I, f, \lambda, r_1, \dots, r_k)$$

where

(1) D and C are sets.

(2) I is a subset of D.

(3) $f: D - I \to C$.

(4) k is a positive integer.

(5) $\lambda: I \times C^k \to C$.

(6) $r_1, \dots, r_k: I \to D$.

D is the *domain* of the recurrence, f is the *base case*, I is the *recursion set*, and the functions $r_1, \dots, r_k$ are the *dependencies*. (C is a mnemonic for *codomain*.)

As with the definition above, we usually say "recurrence" where many authors say "recurrence relation". Other terminology in the definition is also non-standard. Our usage of "multi-" indicates that multiple dependencies are allowed (although the number of dependencies must be finite and constant.) Multi-recurrences are usually described by equations defining their solutions instead of their representation as tuples.

Definition. A *solution* of a multi-recurrence

$$(D, C, I, f, \lambda, r_1, \dots, r_k)$$

is a function $T: D \to C$ that satisfies $T|_{D-I} = f$ and

$$T(x) = \lambda\left(x, T\big(r_1(x)\big), \dots, T\big(r_k(x)\big)\right)$$

for all $x \in I$.

If a multi-recurrence

$$(D, C, I, f, \lambda, r_1, \dots, r_k)$$

has $I = \phi$, then the domain of the recurrence is the domain of the base case; there is no recursion, the base case is the unique solution, and $\lambda, r_1, \dots, r_k$ are the empty function. If $I = D$, then the base case is the empty function. If $I = D \neq \phi$, then recursion is infinite. (Finite and infinite recursion are defined in Section 8.) If $D = \phi$, then the empty function is the only solution of the recurrence.

Multi-recurrence for Fibonacci numbers. Let

$$I = \{n \in \boldsymbol{Z} : n > 2\}.$$

Define $f: \{1,2\} \to \boldsymbol{Z}^+$ by $f(1) = f(2) = 1$, and let

$$\lambda: I \times \boldsymbol{Z}^+ \times \boldsymbol{Z}^+ \to \boldsymbol{Z}^+$$

be the function defined by $\lambda(x, y, z) = y + z$. Define $r_1, r_2: I \to \boldsymbol{Z}^+$ by $r_1(n) = n - 1$ and $r_2(n) = n - 2$. A solution of the multi-recurrence

$$(\boldsymbol{Z}^+, \boldsymbol{Z}^+, I, f, \lambda, r_1, r_2),$$

is a function $T: \boldsymbol{Z}^+ \to \boldsymbol{Z}^+$ that satisfies

$$T(1) = T(2) = 1$$

and

$$T(n) = \lambda\Big(n, T\big(r_1(n)\big), T\big(r_2(n)\big)\Big) = T(n-1) + T(n-2),$$

for all $n > 2$. The function that maps each positive integer n to the nth Fibonacci number is the unique solution of the recurrence.

Multi-recurrence for binomial coefficients. The well-known recurrence relation for binomial coefficients is

$$\binom{n}{0} = \binom{n}{n} = 1$$

for all integers $n \geq 0$, and

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

for all integers satisfying $0 < k < n$. The recurrence relation may be regarded as a multi-recurrence. Let

$$D = \{(n,k) \in \boldsymbol{Z} \times \boldsymbol{Z} : 0 \leq k \leq n\}$$

and

$$I = \{(n,k) \in D : 0 < k < n\}.$$

Define $f: D - I \to \boldsymbol{Z}^+$ by $f(n,k) = 1$ and define $r_1, r_2: I \to \boldsymbol{Z}^+$ by

$$r_1(n,k) = (n-1, k-1)$$

and

$$r_2(n,k) = (n-1, k).$$

Define

$$\lambda: I \times \boldsymbol{Z}^+ \times \boldsymbol{Z}^+ \to \boldsymbol{Z}^+$$

by

$$\lambda(x,y,z) = y + z.$$

The unique solution of the multi-recurrence

$$(D, \boldsymbol{Z}^+, I, f, \lambda, r_1, r_2)$$

is given by

$$T(n,k) = \binom{n}{k}.$$

Example of a multi-recurrence with no solution. Let $I = \{n \in \boldsymbol{Z} : n > 1\}$, and define $f: \{1\} \to \boldsymbol{Z}^+$ by $f(1) = 1$. Define $\lambda: I \times \boldsymbol{Z}^+ \to \boldsymbol{Z}^+$ by $\lambda(m,n) = q(n)$, where $q: \boldsymbol{Z}^+ \to \boldsymbol{Z}^+$ is defined by

$$q(n) = \begin{cases} n+1, & \text{for } n \text{ odd} \\ n-1, & \text{for } n \text{ even,} \end{cases}$$

i.e.

$$1 \mapsto 2 \mapsto 1,$$
$$3 \mapsto 4 \mapsto 3,$$
$$5 \mapsto 6 \mapsto 5,$$
$$\ldots$$

Define $r: I \to I$ by

$$r(n) = \begin{cases} n+1, & \text{for } n \equiv -1 \text{ or } 0 \pmod 3 \\ n-2, & \text{for } n \equiv 1 \pmod 3, \end{cases}$$

i.e.,

$$2 \mapsto 3 \mapsto 4 \mapsto 2$$
$$5 \mapsto 6 \mapsto 7 \mapsto 5,$$
$$8 \mapsto 9 \mapsto 10 \mapsto 8,$$
$$\ldots$$

Observe that q^2 and r^3 are the identify maps on $\boldsymbol{Z}^+$ and I, respectively. (Powers of q and r represent composition of functions.) Suppose T is a solution of the multi-recurrence

$$(\boldsymbol{Z}^+, \boldsymbol{Z}^+, I, f, \lambda, r).$$

If $n \in I$, then $r(n), r^2(n) \in I$ and

$$T(n) = q\left(T\big(r(n)\big)\right) = q^2\left(T\big(r^2(n)\big)\right) = T\big(r^2(n)\big) = q\left(T\big(r^3(n)\big)\right) = q\big(T(n)\big).$$

Since q has no fixed points, there is no such solution T.

Examples of multi-recurrences with infinitely many solutions. Define $r: \boldsymbol{Z} \to \boldsymbol{Z}$ by $r(n) = n - 1$. Let $\lambda: \boldsymbol{Z} \times \boldsymbol{Z} \to \boldsymbol{Z}$ be the projection onto the second component, i.e., $\lambda(n, t) = t$ for all integers n and t. A solution of the multi-recurrence

$$R = (\boldsymbol{Z}, \boldsymbol{Z}, \boldsymbol{Z}, \emptyset, \lambda, r)$$

is any function $T: \boldsymbol{Z} \to \boldsymbol{Z}$ that satisfies

$$T(n) = \lambda\left(n, T\big(r(n)\big)\right) = T\big(r(n)\big) = T(n-1)$$

for all $n \in \boldsymbol{Z}$. For each integer k, there is a solution T_k of the recurrence defined by $T_k(n) = k$ for all n. Indeed, $\{T_k : k \in \boldsymbol{Z}\}$ is the set of all solutions of R. Furthermore, $T_i \neq T_j$ whenever $i \neq j$. Therefore, the recurrence has an infinite number of solutions.

The recurrence has an empty base case. However, a slightly modified version has a non-empty base case: Let $D = \boldsymbol{Z} \cup B$, where B is any non-empty set that does not contain any integers. Let f be any function from B to $\boldsymbol{Z}$. (Such functions exist. For example, let

$b \mapsto 0$ for all $b \in B$.) Each solution T_k of R can be extended to a solution T_k^* of the multi-recurrence

$$S = (\boldsymbol{D}, \boldsymbol{Z}, \boldsymbol{Z}, f, \lambda, r)$$

defined by $T_k^*|_{\boldsymbol{Z}} = T_k$ and $T_k^*|_B = f$. The solutions T_i^* and T_j^* disagree on $\boldsymbol{Z}$ whenever $i \neq j$, so S has an infinite number of solutions.

Existence and uniqueness of solution. In Section 8, we shall show (Lemma 8.2) that every *finitely recursive* multi-recurrence has a unique solution.

Definition. A *semi-divide-and-conquer recurrence* is a $(3k + 4)$-tuple

$$R = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k)$$

where

(1) D is a set of real numbers.

(2) I is a non-empty upper subset of D with a positive lower bound.

(3) k is any positive integer.

(4) $a_1, \dots, a_k$ are positive real numbers.

(5) $b_1, \dots, b_k$ are real numbers satisfying $0 < b_i < 1$ for all $i \in \{1, \dots, k\}$.

(6) f is a real-valued function on $D - I$ with a positive lower bound and finite upper bound.

(7) g is a non-negative real-valued function on I.

(8) $h_1, \dots, h_k$ are real-valued functions on I.

(9) $b_i x + h_i(x) \in D$ for all $x \in I$ and all $i \in \{1, \dots, k\}$.

R is *proper* if $b_i x + h_i(x) < x$ for all $x \in I$ and all $i \in \{1, \dots, k\}$; otherwise R is *improper*. The set D is the *domain* of the recurrence, f is the *base case*, I is the *recursion set*, and g is the *incremental cost*. The functions $h_1, \dots, h_k$ are the *noise terms* of the recurrence, and the functions $b_1 x + h_1(x), \dots, b_k x + h_k(x)$ on I are the *dependencies*. The recursion set is also called the *recursion interval* when it is an interval.

Definition. A *divide-and-conquer recurrence* is a proper semi-divide-and-conquer recurrence. A *mock divide-and-conquer recurrence* is an improper semi-divide-and-conquer *recurrence*.

We are primarily interested in divide-and-conquer recurrences. However, mock divide-and-conquer recurrences also arise in our analysis of Leighton's Theorem 2. Furthermore, two of our main results (see Section 20) are applicable to both divide-and-conquer recurrences and mock divide-and-conquer recurrences.

Semi-divide-and-conquer recurrences are usually described by equations defining their solutions instead of their representation as tuples.

Definition. A solution of a semi-divide-and-conquer recurrence

$$(D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k)$$

is a real-valued function T on D such that $T|_{D-I} = f$ and

$$T(x) = \sum_{i=1}^{k} a_i T\big(b_i x + h_i(x)\big) + g(x)$$

for all $x \in I$.

Representation as multi-recurrences. A semi-divide-and-conquer recurrence

$$(D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k)$$

can be regarded as the multi-recurrence

$$(D, \boldsymbol{R}, I, f, \lambda, r_1, \dots, r_k)$$

where each function $r_i: I \to D$ is defined by $r_i(x) = b_i x + h_i(x)$, and $\lambda: I \times \boldsymbol{R}^k \to \boldsymbol{R}$ is defined by

$$\lambda(x, z_1, \dots, z_k) = \sum_{i=1}^{k} a_i z_i + g(x).$$

A semi-divide-and-conquer recurrence and its corresponding multi-recurrence have the same domain, base case, recursion set, dependencies, and solutions.

Unlike our definition of a multi-recurrence, we require a semi-divide-and-conquer recurrence to have a non-empty recursion set. Multi-recurrences are allowed to have empty recursion sets as a minor convenience for our discussion of depth of recursion in Section 8.

Like multi-recurrences, semi-divide-and-conquer recurrences are allowed to have empty base cases. (The empty function trivially satisfies the requirement for a positive lower bound and finite upper bound.) Of course, an empty base case for a semi-divide-and-conquer recurrence implies infinite recursion since the recursion set is non-empty. Perhaps surprisingly, our most interesting proposition about solutions of semi-divide-and-conquer recurrences does not require a non-empty base case: A solution T of an *admissible recurrence* R satisfies the *strong Akra-Bazzi condition* relative to R and g for each *tame* extension g of the incremental cost of R (equivalently, relative to R and one such g) if and only if T is locally $\Theta(1)$. See Section 20 for further details.

Representation of the dependencies as $\boldsymbol{b_i x + h_i(x)}$. The definition of a semi-divide-and-conquer recurrence includes representation of the dependencies as functions of the form $x \mapsto b_i x + h_i(x)$ where $0 < b_i < 1$ and h_i is a real-valued function on the recursion set I. This is an illusory requirement: Given any real number b and any real-valued function r on I, we have $r(x) = bx + h(x)$ for $h: I \to \boldsymbol{R}$ defined by $h(x) = r(x) - bx$. In particular, the representation is not unique: Let

$$(D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k)$$

be any semi-divide-and-conquer recurrence. Given $b_1^*, \dots, b_k^* \in (0,1)$, define real-valued functions $h_1^*, \dots, h_k^*$ on I by $h_i^*(x) = (b_i - b_i^*)x + h_i(x)$, so that

$$b_i x + h_i(x) = b_i^* x + h_i^*(x).$$

Then

$$(D, I, a_1, \dots, a_k, b_1^*, \dots, b_k^*, f, g, h_1^*, \dots, h_k^*)$$

is also a semi-divide-and-conquer recurrence. Furthermore, the two recurrences represent the same multi-recurrence and have the same solutions. Either both recurrences are proper or both are improper.

Now suppose I is unbounded and

$$\lim_{x\to\infty} \frac{h_i(x)}{x} = \lim_{x\to\infty} \frac{h_i^*(x)}{x} = 0.$$

Observe that

$$b_i = \lim_{x\to\infty} \left(b_i + \frac{h_i(x)}{x} \right) = \lim_{x\to\infty} \left(b_i^* + \frac{h_i^*(x)}{x} \right) = b_i^*,$$

i.e., $b_i = b_i^*$, which implies $h_i = h_i^*$.

Our main results (in Sections 20 and 21) apply to semi-divide-and-conquer recurrences with a couple additional properties, including *low noise* (defined in Section 20). If the recursion set is unbounded (the most interesting case), low noise implies

$$\lim_{x\to\infty} \frac{h_i(x)}{x} = 0$$

for all $i \in \{1, \dots, k\}$. Each dependency of such a recurrence has a unique representation of the form $b_i x + h_i(x)$ that is consistent with the definition of low noise.

Floor and ceiling noise. Common examples of dependencies in divide-and-conquer recurrences are functions of the form $x \mapsto \lfloor bx \rfloor$ and $x \mapsto \lceil bx \rceil$ where $b \in (0,1)$. The corresponding noise terms are the functions $x \mapsto \lfloor bx \rfloor - bx$ and $x \mapsto \lceil bx \rceil - bx$.

Dependency graph. The condition $b_i x + h_i(x) < x$ of a divide-and-conquer recurrence implies such a recurrence has no circular dependencies: The directed multigraph with D as its vertex set and with directed edges from all $x \in I$ to the vertices

$$b_1 x + h_1(x), \dots, b_k x + h_k(x)$$

is acyclic.

Requirement that the base case is $\boldsymbol{\Theta(1)}$. Our definition of a semi-divide-and-conquer recurrence specifies the base case to be $\Theta(1)$. A broader and arguably more natural class of recurrences can be obtained by instead simply requiring the base case to be non-negative. However, our most interesting conclusions about divide-and-conquer recurrences require the base case to be $\Theta(1)$. For convenience, we incorporate this restriction directly into our definition.

In practice, the restriction is usually minor. For a multi-recurrence whose domain is a set of integers with a finite lower bound, the base case is $\Theta(1)$ if and only if the base case is positive. Furthermore, roots of a base case are often inessential. For example, the unique solution $T: \boldsymbol{N} \to \boldsymbol{N}$, defined by $T(n) = n$ for all $n \in \boldsymbol{N}$, of the recurrence

$$T(n) = \begin{cases} 0, & \text{for } n = 0 \\ 1, & \text{for } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil), & \text{for } n > 1 \end{cases}$$

is nearly identical to the unique solution $T^*: \boldsymbol{Z}^+ \to \boldsymbol{Z}^+$, defined by $T^*(n) = n$ for all $n \in \boldsymbol{Z}^+$, of the divide-and-conquer recurrence

$$T^*(n) = \begin{cases} 1, & \text{for } n = 1 \\ 2, & \text{for } n = 2 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil), & \text{for } n > 2. \end{cases}$$

The functions T and T^* differ only in the exclusion of 0 from the domain of T^*. In particular, they are asymptotically identical.

In contrast, consider the recurrence

$$S(n) = \begin{cases} 0, & \text{for } n = 1 \\ 2, & \text{for } n = 2 \\ 2 \cdot S(\lfloor n/2 \rfloor), & \text{for } n > 2 \end{cases}$$

defining a function $S: \boldsymbol{Z}^+ \to \boldsymbol{N}$. Observe that $S(2^k \cdot 3) = 0$ for each $k \in \boldsymbol{N}$. Let D be any subset of $\boldsymbol{Z}^+$ with $\boldsymbol{Z}^+ \backslash D$ finite, and let I be a non-empty upper subset of D such that $\lfloor n/2 \rfloor \in D$ for all $n \in I$. Observe that $|D \backslash I| < \min I$. In particular, $D \backslash I$ is finite. The relative complement $\boldsymbol{Z}^{+}\backslash I = (\boldsymbol{Z}^{+}\backslash D) \cup (D\backslash I)$ is a union of two finite sets and is therefore also finite. Define $S^{*}: D \to \boldsymbol{Z}$ by the recurrence

$$S^{*}(n) = \begin{cases} S(n), & \text{for } n \in D\backslash I \\ 2 \cdot S^{*}(\lfloor n/2 \rfloor), & \text{for } n \in I, \end{cases}$$

so $S^{*}(n) = S(n)$ for all $n \in D$. Let $Y = \{2^{j} \cdot 3 : j \in \boldsymbol{N}\}$, so Y is an infinite set of positive integers. The intersection $Y \cap I$ is non-empty because $\boldsymbol{Z}^{+}\backslash I$ is finite. Let y be the least element of $Y \cap I$, so $y = 2^{m} \cdot 3$ for some $m \in \boldsymbol{N}$. Let $d = \lfloor y/2 \rfloor$, so $d \in D$. We claim $d \notin I$ and d is a root of S^{*}. If $m = 0$, then $y = 3$, so $d = 1$ and $\lfloor d/2 \rfloor = 0 \notin \boldsymbol{Z}^{+}$; therefore, $\lfloor d/2 \rfloor \notin D$, which implies $d \notin I$; furthermore, $S^{*}(d) = S(1) = 0$. If $m > 0$, then $m - 1 \in \boldsymbol{N}$ and $d = 2^{m-1} \cdot 3 \in Y$; then $d \notin I$ because $d < y$; furthermore, $S^{*}(d) = S(d) = 0$. The claim is proved. In particular, the recurrence above defining S^{*} is not a semi-divide-and-conquer recurrence.

Conditional uniqueness and positivity of solutions. As we shall see, a solution of a semi-divide-and-conquer recurrence need not be unique or positive. Section 13 exhibits for each $x_0 \in [686{,}10000]$ an infinitely recursive, semi-divide-and-conquer recurrence with recursion interval (x_0, ∞). (The recurrence is proper only when $x_0 = 10000$.) Each recurrence in the family has infinitely many solutions. Uncountably many solutions of each recurrence surjectively map each non-empty open subset of the recursion interval onto the real line.

Finite recursion significantly constrains the landscape of solutions: Each finitely recursive semi-divide-and-conquer recurrence has a unique solution, which is positive. (Corollary 8.5 adds positivity to the existence and uniqueness established by Lemma 8.2 for finitely recursive multi-recurrences.)

Leighton's recurrrences. Theorems 1 and 2 of [Le] involve recurrences of the form

$$T(x) = \begin{cases} f(x), & \text{for } 1 \le x \le x_0 \\ \displaystyle\sum_{i=1}^{k} a_i T\big(b_i x + h_i(x)\big) + g(x), & \text{for } x > x_0 \end{cases}$$

where $f: [1, x_0] \to \boldsymbol{R}^{+}$ is $\Theta(1)$, g is a non-negative function, h_i is real-valued, each a_i is a positive real number, and each b_i satisfies $0 < b_i < 1$. Theorem 1 omits the $h_i(x)$ term, i.e., each h_i is identically zero. Theorem 2 allows non-zero h_i. There are various restrictions on the value of x_0. In particular, $x_0 > 1$.

Neither the domain of g nor the domain of h_i is specified in [Le]. Recurrences of this form require only that those domains contain (x_0, ∞). However, the function g is required by [Le] to satisfy Leighton's polynomial-growth condition, which implicitly requires $domain(g)$ to contain $[b_i, \infty)$ for each i. Furthermore, condition (2) of Theorem 2 implicitly assumes the domain of h_i contains $[x_0, \infty)$. In addition, condition (3) implicitly assumes the domain of h_i contains $[1, \infty)$ and $domain(g)$ contains the interval $[b_i x + h_i(x), x]$ for all $x \geq 1$. In particular, the domains of g and h_i must properly contain (x_0, ∞).

Let $D = [1, \infty)$ and $I = (x_0, \infty)$. Recurrences that satisfy the hypotheses of Theorem 1 or Theorem 2 are apparently intended to have the property that

$$(D, I, a_1, \ldots, a_k, b_1, \ldots, b_k, f, g|_I, h_1|_I, \ldots, h_k|_I)$$

satisfies our definition of a divide-and-conquer recurrence.

However, the recurrences in Section 13 with $x_0 < 10000$ are mock divide-and-conquer recurrences that satisfy the conditions of Theorem 2. They have $b_1 x + h_1(x) = x$ for $x = 10000$ and $b_1 x + h_1(x) > x$ for all $x \in (x_0, 10000)$.

A recurrence is exhibited in Section 19 that satisfies the hypothesis of Theorem 2 but is not a semi-divide-and-conquer recurrence. Condition (9) of the definition is violated: the recurrence has $b_1 x + h_1(x) < 1$ for infinitely many $x > x_0$, i.e., $b_1 x + h_1(x) \notin D$. In particular, that recurrence has no solution with the specified domain.

Although [Le] requires the domain of g (and h_i in Theorem 2) to properly contain the recursion interval I and constrains the behavior of those functions outside I, the recursive definition of T does not depend on the value of $g(x)$ or $h_i(x)$ for any $x \leq x_0$. Unfortunately, the propositions in [Le] needlessly run aground if g has polynomial growth on (x_0, ∞) but not on some $[b_i, \infty)$. The situation is illustrated by the second example in [Le] as explained in Section 6. Condition 3 of Theorem 2 is similarly problematic.

Replacements for Leighton's Propositions. Our replacement in Section 11 for Leighton's Theorem 1 is applicable to divide-and-conquer recurrences that have zero noise and satisfy a few mild conditions. Our replacements in Section 20 for Leighton's Theorem 2 are applicable to mildly constrained divide-and-conquer recurrences and (with one exception) mock divide-and-conquer recurrences. By definition of a semi-divide-and-conquer recurrence, the domains of the incremental cost g and the noise functions h_i are the recursion set in each of our replacements for Leighton's intended theorems.

The new propositions omit Leighton's polynomial-growth condition, which is incompatible with the new domain of the incremental cost g. We assume instead that g has polynomial growth as defined herein. Condition 2 of Theorem 2 is replaced with an analogous condition, which does not require $domain(h_i)$ to properly contain the recursion set. (We also replace the upper bound for $|h_i|$ with an asymptotic condition.) Condition 3 is eliminated altogether.

8. Depth of Recursion

Relative depth. For each multi-recurrence

$$R = (D, C, I, f, \lambda, r_1, \dots, r_k),$$

with domain D, recursion set I, and dependencies $r_1, \dots, r_k : I \to D$, there is an associated function $u: 2^D \to 2^D$ defined by

$$u(B) = B \cup \{ x \in I : r_i(x) \in B \text{ for each } i \}$$

for all $B \subseteq D$. (2^D is the power set of D, i.e., the set of all subsets of D.) Powers of u represent composition of functions, i.e., u^0 is the identity map on 2^D and $u^n = u \circ u^{n-1}$ for all positive integers n.

The function u preserves set inclusion: If $A \subseteq B \subseteq D$, then $u(A) \subseteq u(B)$, and by induction, $u^n(A) \subseteq u^n(B)$ for all non-negative integers n. By definition, $B \subseteq u(B)$, which implies $u^n(B) \subseteq u^{n+1}(B)$, i.e.,

$$B = u^0(B) \subseteq u(B) \subseteq u^2(B) \subseteq u^3(B) \subseteq \cdots.$$

Observe that

$$u^0(B) = B \subseteq B \cup I.$$

If $m \in \boldsymbol{N}$ such that

$$u^m(B) \subseteq B \cup I,$$

then

$$u^{m+1}(B) \subseteq u(B \cup I) = u(B) \cup u(I) \subseteq (B \cup I) \cup I = B \cup I.$$

By induction, $u^n(B) \subseteq B \cup I$ for all $n \in \boldsymbol{N}$.

For each $B \subseteq D$, define a function $d_B: D \cup 2^D \to \boldsymbol{N} \cup \{\infty\}$ by

$$d_B(x) = \begin{cases} 0, & \text{if } x \in B \\ n \in \boldsymbol{Z}^+, & \text{if } x \in u^n(B) \text{ and } x \notin u^{n-1}(B) \\ \infty, & \text{if } x \notin u^n(B) \text{ for all } n \in \boldsymbol{N} \end{cases}$$

for all $x \in D$,

$$d_B(S) = \sup\{d_B(x) : x \in S\}$$

when $\phi \neq S \subseteq D$, and $d_B(\phi) = 0$. Observe that

$$d_B(x) = 1 + \max_{1 \le i \le k} d_B\big(r_i(x)\big)$$

for all $x \in I - B$. (If $x \in I$ with $d_B(x) = \infty$, then $d_B\big(r_i(x)\big) = \infty$ for some i, and the equation above is $\infty = 1 + \infty$.) We have $d_B(x) \in \{0, \infty\}$ for all $x \in D - I$ because

$$u^n(B) \cap (D - I) \subseteq (B \cup I) \cap (D - I) = B - I \subseteq B = u^0(B)$$

for all $n \in \boldsymbol{N}$.

If there exists $t \in D$ with $0 < d_B(t) < \infty$, then $t \in I - B$, and $d_B(t) = 1 + d_B\big(r_i(t)\big)$ for some index i. Given the existence of t, a simple inductive argument shows that for each integer m satisfying $0 \le m \le d_B(t)$, there exists $z \in D$ with $d_B(z) = m$.

The function d_B is monotonically increasing relative to set inclusion: If $V \subseteq W \subseteq D$, then $d_B(V) \le d_B(W)$. Similarly, $d_B(X \cup Y) = \max\big(d_B(X), d_B(Y)\big)$ for all $X, Y \subseteq D$.

For each $x \in D$, the quantity $d_B(x)$ is called the recurrence's (maximum) *depth of recursion at x relative to B*. Let $S \subseteq D$. If $d_B(S) < \infty$, then there exists $y \in S$ with $d_B(S) = d_B(y)$, and $d_B(S)$ is called the *maximum depth of recursion on S relative to B*, and we say the recurrence has *bounded depth of recursion on S relative to B*; otherwise, the recurrence has *unbounded depth of recursion on S relative to B*.

The terminology here may seem upside down to some readers. The quantity $d_B(x)$ might be regarded instead as the *height* of x above B relative to the recurrence. Nonetheless, we shall continue in our use of "depth".

We are interested in depth of recursion relative to a subset B of D only when $D - I \subseteq B$ and $r_i(B \cap I) \subseteq B$ for each i. Otherwise, "depth of recursion relative to B" is a misnomer.

We sometimes use $d(x)$ and $d(S)$ as shorthand for $d_{D-I}(x)$ and $d_{D-I}(S)$ respectively where $x \in D$ and $S \subseteq D$. Without further qualification, depth of recursion is relative to $D - I$, the domain of the base case. The recurrence is *finitely recursive* if $d(x) < \infty$ for all $x \in D$; otherwise the recurrence is *infinitely recursive*. Of course, an empty base case implies infinite depth of recursion at each element of D.

The depth-of-recursion functions d and d_B are determined by the choice of recurrence. The recurrence associated with a particular reference to d or d_B should be clear from context.

Suppose T is a solution of the multi-recurrence R. For $x \in D$ with $d(x) < \infty$, the quantity $d(x)$ is roughly proportional to the maximum call stack height during the recursive computation of $T(x)$ provided the computation of f, λ, and $r_1, \dots, r_k$ is non-recursive. For $S \subseteq D$ with $d(S) < \infty$, the quantity $d(S)$ is similarly related to the maximum call stack height for the restriction of T to S. Of course, stack overflow is associated with $d(x) = \infty$ and $d(S) = \infty$.

The depth-of-recursion function d for the multi-recurrence R satisfies the following recurrence:

$$d(x) = \begin{cases} 0, & \text{for } x \in D - I \\ 1 + \max\limits_{1 \le i \le k} d\big(r_i(x)\big), & \text{for } x \in I. \end{cases}$$

Of course d is also the depth-of-recursion for this auxiliary recurrence.

(If $k = 1$ and R represents a semi-divide-and-conquer recurrence, then the recurrence satisfied by d resembles a semi-divide-and-conquer recurrence, but the definition of such a recurrence is violated by $range\big(d|_{D \setminus I}\big) = \{0\}$.

Lemma 8.1. Suppose

$$(D, C, I, f, \lambda, r_1, \dots, r_k)$$

is a multi-recurrence. Define $E_n = \{x \in D : d(x) \le n\}$, $I_n = I \cap E_n$, and

$$R_n = \big(E_n, C, I_n, f, \lambda|_{I_n \times C^k}, r_1|_{I_n}, \dots, r_k|_{I_n}\big)$$

for each non-negative integer n. Then R_n is a multi-recurrence with a unique solution T_n. Furthermore,

$$T_{n+1}|_{E_n} = T_n.$$

Proof. (Of course, $E_n = u^n(D - I)$.) The set I_n is obviously contained in E_n as required by the definition of a multi-recurrence. The identity $D - I = E_0 = \{x \in D : d(x) = 0\}$ implies $I = \{x \in D : d(x) \ge 1\}$ and $I_n = \{x \in D : 1 \le d(x) \le n\}$. (Notice that $I_0 = \emptyset$.) We conclude that $E_n - I_n = D - I$. In particular, $E_n - I_n$ is the domain of f.

By definition of a multi-recurrence, the function λ maps $I \times C^k$ to C. The set containment $I_n \subseteq I$ implies $I_n \times C^k \subseteq I \times C^k$, so the restriction of λ to $I_n \times C^k$ is defined and maps $I_n \times C^k$ to C as required by the definition of a multi-recurrence. Finally,

$$I_n \subseteq I = domain(r_i)$$

and $r_i(I_n) \subseteq E_n$ for all $i \in \{1, \dots, k\}$, i.e., the restriction of each r_i to E_n is defined and maps I_n to E_n as required. Therefore, R_n is a multi-recurrence.

Since $E_0 = D - I$, the function $T_0 = f$ is the unique solution of R_0. Let n be a non-negative integer such that R_n has a unique solution T_n. Since $E_n \subseteq E_{n+1}$ and

$$r_1(x), \dots, r_k(x) \in E_n$$

for all $x \in E_{n+1} - E_n$, the multi-recurrence R_{n+1} has the unique solution $T_{n+1}: E_{n+1} \to C$ defined by $T_{n+1}|_{E_n} = T_n$ and

$$T_{n+1}(x) = \lambda\left(x, T_n\big(r_1(x)\big), \dots, T_n\big(r_k(x)\big)\right)$$

for all $x \in E_{n+1} - E_n = I_{n+1} - I_n$. The lemma follows by induction. □

Lemma 8.2. Every finitely recursive multi-recurrence has a unique solution.

Proof. Let

$$R = (D, C, I, f, \lambda, r_1, \dots, r_k)$$

be a finitely recursive multi-recurrence. Define E_n, R_n, and T_n as in Lemma 8.1. Finite recursion implies

$$D = \bigcup_{n \geq 0} E_n.$$

A function $T: D \to C$ is a solution of R if and only if $T|_{E_n}$ is a solution of R_n for each non-negative integer n, i.e., $T|_{E_n} = T_n$ for all such n. Furthermore, there can be at most one such function.

Suppose α is a non-negative integer. We have the trivial identity $T_\alpha|_{E_\alpha} = T_\alpha$. If β is an integer that satisfies $\beta \geq \alpha$ and $T_\beta|_{E_\alpha} = T_\alpha$, then

$$T_{\beta+1}|_{E_\alpha} = \left(T_{\beta+1}|_{E_\beta}\right)|_{E_\alpha} = T_\beta|_{E_\alpha} = T_\alpha.$$

By induction, $T_\gamma|_{E_\alpha} = T_\alpha$ for all integers $\gamma \geq \alpha$.

Define T by $T(x) = T_{d(x)}(x)$ for all $x \in D$. If $w \in E_n$, then $d(w) \leq n$, so

$$T(w) = T_{d(w)}(w) = T_n|_{E_{d(w)}}(w) = T_n(w).$$

Therefore, $T|_{E_n} = T_n$ as required. □

Lemma 8.3. If D is the domain of a multi-recurrence, and $A \subseteq B \subseteq D$, then

$$d_B(x) \leq d_A(x) \leq d_B(x) + d_A(B)$$

for all $x \in D$ and

$$d_B(S) \leq d_A(S) \leq d_B(S) + d_A(B)$$

for all $S \subseteq D$. In particular, if the recurrence has bounded depth of recursion on S and B relative to B and A, respectively, then the recurrence has bounded depth of recursion on S relative to A.

Proof. Since $u^n(A) \subseteq u^n(B)$ for each non-negative integer n, we conclude that $d_B(x) \leq d_A(x)$ for all $x \in D$. If $d_B(x) < \infty$ and $d_A(B) < \infty$, then $A \neq \phi$, $B \neq \phi$, and

$$x \in u^{d_B(x)}(B) \subseteq u^{d_B(x)}\left(u^{d_A(B)}(A)\right) = u^{d_B(x)+d_A(B)}(A),$$

so

$$d_A(x) \leq d_B(x) + d_A(B).$$

The preceding inequality is also valid if $d_B(x) = \infty$ or $d_A(B) = \infty$. Combining inequalities, we obtain

$$d_B(x) \leq d_A(x) \leq d_B(x) + d_A(B)$$

for all $x \in D$, so

$$d_B(S) \leq d_A(S) \leq d_B(S) + d_A(B)$$

for all $S \subseteq D$. (The inequality above is $0 \leq 0 \leq 0 + d_A(B)$ if $S = \phi$.) The remaining assertion follows from the final inequality above and the definition of bounded depth of recursion. □

We now turn our attention to divide-and-conquer recurrences and (temporarily) mock divide-and-conquer recurrences. Our discussion about depth of recursion is applicable to them via their representations as multi-recurrences. We shall provide simple, naive, crude bounds to solutions of many such recurrences in terms of depth of recursion. Of course, more sophisticated bounds for a large class of such recurrences will be provided in later sections.

Notation. For the remainder of this section, T is a (not necessarily unique) solution of a semi-divide-and-conquer recurrence

$$R = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k)$$

with domain D, recursion set I, base case f, and incremental cost g. (Starting with Lemma 8.6, we assume R is proper, i.e., is a divide-and-conquer recurrence.) We do not assume finite recursion. Let $r_1, \dots, r_k: I \to D$ be the dependencies of R, i.e.,

$$r_i(x) = b_i x + h_i(x)$$

for all $x \in I$.

Define $y = \inf f$ and $Y = \sup f$, i.e., $y = \inf T(D \setminus I)$ and $Y = \sup T(D \setminus I)$ if $D \neq I$, i.e., R has a non-empty base case, i.e., R is finitely recursive at some element of D. Define $y = 1$ and $Y = 1$ if $D = I$, i.e., R has an empty base case, i.e., R is infinitely recursive at every element of D. Observe that y and Y are lower and upper bounds, respectively, for

f, which is $T|_{D\setminus I}$, regardless of whether $D = I$ or $D \neq I$. By definition of a semi-divide-and-conquer recurrence,

$$0 \leq y \leq Y < \infty.$$

Define

$$A = \sum_{i=1}^{k} a_i,$$

so $A > 0$. Let $U = \max(A, 1)$. As usual, d is the depth-of-recursion function for R relative to $D\setminus I$.

Define the function $G: D \to [0, +\infty]$ by

$$G(x) = \begin{cases} \sup g(I \cap (0, x]), & \text{for } x \in I \\ 0, & \text{for } x \in D\setminus I. \end{cases}$$

The function G is monotonically increasing, and $0 \leq g(x) \leq G(x)$ for each x in I. If g is bounded on each bounded subset of I, then the function G is real-valued, i.e., $G(x) < \infty$ for all $x \in D$. (The converse is true if I is unbounded or has a maximum element.) If I contains only integers, then each bounded subset of I is finite, so g is bounded on bounded sets and G is real-valued. Alternatively, if g has polynomial growth, then Corollary 2.23 combines with $\inf I > 0$ to imply g is bounded on bounded sets and G is real valued.

Lemma 8.4.

$$\inf T|_{E_n} \geq y \cdot \min\{1, A^n\} > 0$$

for each non-negative integer n where

$$E_n = \{x \in D : d(x) \leq n\}.$$

Proof. Observe that

$$y \cdot \min\{1, A^0\} = y$$

is a positive lower bound of $T|_{E_0}$ because $E_0 = D\setminus I$ and $A^0 = 1$.

Suppose m is a non-negative integer such that

$$\inf T|_{E_m} \geq y \cdot \min\{1, A^m\} > 0.$$

The set $E_{m+1}\setminus E_m$ is contained in the recursion set, I, and $r_i(E_{m+1}\setminus E_m)$ is contained in E_m for all $i \in \{1, \ldots, k\}$. Since $a_1, \ldots, a_k$ and A are positive and the incremental cost, g, is non-negative, we conclude that

$$T(z) = \sum_{i=1}^{k} a_i T(r_i(z)) + g(z) \geq \left(\sum_{i=1}^{k} a_i\right) \cdot y \cdot \min\{1, A^m\} = Ay \cdot \min\{1, A^m\}$$

$$= y \cdot \min\{A, A^{m+1}\}$$

for all $z \in E_{m+1} \backslash E_m$. (Of course, $E_{m+1} \backslash E_m$ may be empty, in which case there is no such z.) Therefore,

$$\inf T|_{E_{m+1}} \geq y\alpha$$

where

$$\alpha = \min\{1, A, A^m, A^{m+1}\}.$$

If $A \geq 1$, then

$$\alpha = 1 = \min\{1, A^{m+1}\}.$$

If $A < 1$, then

$$\alpha = A^{m+1} = \min\{1, A^{m+1}\}.$$

Therefore,

$$\inf T|_{E_{m+1}} \geq y \cdot \min\{1, A^{m+1}\} > 0.$$

The lemma follows by induction. □

Corollary 8.5. Every finitely recursive semi-divide-and-conquer recurrence has a unique solution, which is positive.

Proof. The proposition follows from Lemmas 8.2 and 8.4. □

Lemma 8.6. Assume R is proper. If F is an initial subset of D and g is bounded on $F \cap I$, then $T|_{F_n}$ has a finite upper bound for each non-negative integer n where

$$F_n = \{x \in F : d(x) \leq n\}.$$

Proof. Let W be a finite upper bound for the restriction of g to $F \cap I$, and let $E = D \backslash I$, i.e., E is the domain of the recurrence's base case, f. The function f is $T|_E$ and has a finite upper bound by definition of a divide-and-conquer recurrence. The set F_0 is contained in E, so $T|_{F_0}$ also has a finite upper bound.

Suppose m is a non-negative integer for which the restriction of T to F_m has a finite upper bound S. Observe that $F_{m+1} \backslash F_m$ is contained in the recursion set, I. Furthermore, $r_i(F_{m+1} \backslash F_m)$ is contained in F_m for all $i \in \{1, \ldots, k\}$ because the recurrence is proper and F is an initial subset of D. By definition,

$$T(z) = \sum_{i=1}^{k} a_i T(r_i(z)) + g(z)$$

for all $z \in F_{m+1} \backslash F_m$. (Of course, $F_{m+1} \backslash F_m$ may be empty, in which case there is no such z.) Since $a_1, \ldots, a_k$ are positive and $T(r_i(z))) \leq S$ for each index i,

$$T(z) \leq \sum_{i=1}^{k} a_i S + g(z) \leq AS + W < \infty$$

for each such z. The maximum of S and $AS + W$ is a finite upper bound for the restriction of T to F_{m+1}. The lemma follows by induction. □

Corollary 8.7. Assume R is proper. If F is an initial subset of D with $\sup F < \infty$, and g has polynomial growth on $F \cap I$, then $T|_{F_n}$ has a finite upper bound for each non-negative integer n where

$$F_n = \{x \in F : d(x) \leq n\}.$$

Proof. Of course,

$$\sup(F \cap I) \leq \sup F < \infty$$

and

$$\inf(F \cap I) \geq \inf I > 0.$$

Corollary 2.23 implies the polynomial-growth function $g|_{F \cap I}$ is bounded, i.e., g is bounded on $F \cap I$. The proposition follows from Lemma 8.6. □

We now obtain a more explicit upper bound.

Lemma 8.8. Assume R is proper. If $x \in D$ such that $d(x) < \infty$ and $G(x) < \infty$, then

$$T(x) \;\leq\; U^{\delta} Y + G(x) \cdot \sum_{j=0}^{\delta - 1} A^j < \infty$$

where $\delta = d(x)$. (The sum is interpreted as zero when $\delta = 0$.)

Proof. Finiteness of $G(x)$ combines with $\delta \in \boldsymbol{N}$ and $A, U, Y \in \boldsymbol{R}^+$ to imply the finiteness assertion of the lemma, i.e., the rightmost inequality in the asserted chain of inequalities.

We prove the remaining inequality by induction on the depth of recursion. If $\delta = 0$, then the sum is zero, $x \in D \backslash I$, $U^{\delta} = 1$, and $G(x) = 0$. Then

$$T(x) \;\leq\; Y = \; U^{\delta} Y + G(x) \cdot \sum_{j=0}^{\delta - 1} A^j$$

as required.

Now suppose $\delta > 0$ and

$$T(z) \;\leq\; U^{d(z)} Y + G(z) \cdot \sum_{j=0}^{d(z)-1} A^j$$

for all $z \in D$ with $d(z) < \delta$ and $G(z) < \infty$. Positivity of δ implies $x \in I$. Define $\delta_i = d(r_i(x))$ for all $i \in \{1, \dots, k\}$, so $\delta_i \leq \delta - 1$. By definition,

$$T(x) = \sum_{i=1}^{k} a_i T(r_i(x)) + g(x).$$

By definition of a divide-and-conquer recurrence, $r_i(x) < x$ for each index i. The function G is non-negative and monotonically increasing, so

$$0 \leq G(r_i(x) \leq G(x) < \infty$$

for each i. The inductive hypothesis combines with $g(x) \leq G(x)$, positivity of $a_1, \dots, a_k$ and finiteness of $G(r_1(x), \dots, G(r_k(x)$ to imply

$$T(x) \leq \sum_{i=1}^{k} \left(a_i \cdot \left(U^{\delta_i} Y + G\big(r_i(x)\big) \cdot \sum_{j=0}^{\delta_i - 1} A^j \right) \right) + G(x).$$

Observe that

$$a_i G\big(r_i(x)\big) \cdot \sum_{j=0}^{\delta_i - 1} A^j \leq a_i G(x) \cdot \sum_{j=0}^{\delta - 2} A^j$$

for all that $i \in \{1, \dots, k\}$ since $a_i, A \in \boldsymbol{R}^+$ and $\delta_i - 1 \leq \delta - 2$ for each index i.

We also have $a_i U^{\delta_i} Y \leq a_i U^{\delta - 1} Y$ because a_i and Y are positive, $U \geq 1$ and $\delta_i \leq \delta - 1$. Therefore,

$$T(x) \leq \sum_{i=1}^{k} \left(a_i \cdot \left(U^{\delta - 1} Y + G(x) \cdot \sum_{j=0}^{\delta - 2} A^j \right) \right) + G(x)$$

$$= A \cdot \left(U^{\delta - 1} Y + G(x) \cdot \sum_{j=0}^{\delta - 2} A^j \right) + G(x)$$

$$= A U^{\delta - 1} Y + G(x) \cdot \sum_{j=0}^{\delta - 1} A^j.$$

Since $0 < A \leq U$ and $Y > 0$, we have $A U^{\delta - 1} Y \leq U^{\delta} Y$, so

$$T(x) \ \leq U^{\delta} Y + G(x) \cdot \sum_{j=0}^{\delta-1} A^j.$$

The lemma follows by induction. □

Since $A = 1$ for some divide-and-conquer recurrences, the expression

$$\sum_{j=0}^{\delta-1} A^j$$

cannot be replaced by $\left(A^{\delta} - 1\right)/(A - 1)$ in Lemma 8.8.

$\boldsymbol{M_F}$**.** Suppose F is a non-empty initial subset of D, and g is bounded on $F \cap I$, i.e., $\sup G(F) < \infty$. Since G is non-negative and F is non-empty, we conclude that $\sup G(F) \geq 0$, i.e., $\sup G(F)$ is a non-negative real number. For each such F, define a real-valued function $M_F : \boldsymbol{N} \to \boldsymbol{R}$ on the non-negative integers by

$$M_F(n) = U^n Y + (\sup G(F)) \cdot \sum_{j=0}^{n-1} A^j.$$

When $n = 0$, the sum is interpreted as zero, so $M_F(0) = U^0 Y = Y$. The function M_F is positive and monotonically increasing because A and Y are positive, $U \geq 1$, and $\sup G(F) \geq 0$. As a minor convenience, we also define the positive, constant, function $M_\phi(n) = 1$ on the non-negative integers; of course, M_ϕ is also monotonically increasing.

The following simple adaptations of Lemma 8.8 provide a more uniform upper bound for $T(x)$ when g is suitably constrained:

Corollary 8.9. Assume R is proper and let n be a non-negative integer. If F is an initial subset of D, and g is bounded on $F \cap I$, then

$$T(x) \leq M_F(n)$$

for all $x \in F$ that satisfy $d(x) \leq n$.

Proof. The function M_F is monotonically increasing, so Lemma 8.8 combines with

$$G(x) \leq \sup G(F) < \infty$$

and

$$\sum_{j=0}^{d(x)-1} A^j \geq 0$$

to imply

$$T(x) \leq M_F\big(d(x)\big) \leq M_F(n)$$

for all such x. □

Corollary 8.10. Assume R is proper and let n be a non-negative integer. If F is an initial subset of D with $\sup F < \infty$, and g has polynomial growth on $F \cap I$, then

$$T(x) \leq M_F(n)$$

for all $x \in F$ that satisfy $d(x) \leq n$.

Proof. The proof is the same as for Corollary 8.7, except with Corollary 8.9 playing the role of Lemma 8.6:

$$\sup(F \cap I) \leq \sup F < \infty$$

and

$$\inf(F \cap I) \geq \inf I > 0.$$

Corollary 2.23 implies the polynomial-growth function $g|_{F \cap I}$ is bounded, i.e., g is bounded on $F \cap I$. The proposition follows from Corollary 8.9. □

Corollaries 8.9 and 8.10 provide explicit finite upper bounds for T under the assumptions of Lemma 8.6 and Corollary 8.7, respectively. The bounds were derived independently of Lemmas 8.6 and Corollary 8.7, thereby making those propositions redundant (but instructive).

9. Locally $\Theta(1)$ Solutions

Our main propositions about semi-divide-and-conquer recurrences require solutions that are locally $\Theta(1)$. We start with a convenient, obvious characterization of such solutions:

Lemma 9.1. Let T be a solution of a semi-divide-and-conquer recurrence with domain D and recursion set I. The following statements are equivalent:

(1) T is locally $\Theta(1)$.

(2) $T|_I$ is locally $\Theta(1)$.

(3) $T|_Y = \Theta(1)$ for every subset Y of D with $\sup Y < \infty$.

Proof. (1) $\Longrightarrow$ (2): By definition, I is a subset of D. If X is a bounded subset of I, then X is also a bounded subset of D, so $T|_X = \Theta(1)$ by definition of a locally $\Theta(1)$ function.

(2) $\Longrightarrow$ (3): If $Y \subseteq D$ such that $\sup Y < \infty$, then $\sup(Y \cap I) \leq \sup Y < \infty$ and $\inf(Y \cap I) \geq \inf I > 0$, so $Y \cap I$ is bounded, which implies $T|_{Y \cap I} = \Theta(1)$. By definition of a solution of a semi-divide-and-conquer recurrence, $T|_{D-I} = \Theta(1)$. We conclude from $Y - I \subseteq D - I$ that $T|_{Y-I} = \Theta(1)$. The identity $Y = (Y \cap I) \cup (Y - I)$ implies $T|_Y = \Theta(1)$.

(3) $\Longrightarrow$ (1): If S is a bounded subset of D, then $\sup S < \infty$, which implies $T|_S = \Theta(1)$. Therefore T is locally $\Theta(1)$. □

Definition. A semi-divide-and-conquer recurrence satisfies the *bounded depth condition* if $d(S) < \infty$ for every bounded subset S of the recurrence's domain (where d is the depth-of-recursion function for the recurrence).

Of course, satisfaction of the bounded depth condition implies finite recursion. In the same spirit as Lemma 9.1, we identify obviously equivalent formulations of the bounded depth condition:

Lemma 9.2. Let R be a semi-divide-and-conquer recurrence with domain D and recursion set I. The following statements are equivalent:

(1) R satisfies the bounded depth condition.

(2) $d(X) < \infty$ for every bounded subset X of I.

(3) $d(Y) < \infty$ for every subset Y of D with $\sup Y < \infty$.

Proof. Let d be the depth-of-recursion function. (1) ⟹ (2): By definition, I is a subset of D. If X is a bounded subset of I, then X is also a bounded subset of D, so $d(X) < \infty$ by definition of the bounded depth condition.

(2) ⟹ (3): If Y is a subset of D such that $\sup Y < \infty$, then $\sup(Y \cap I) \leq \sup Y < \infty$ and $\inf(Y \cap I) \geq \inf I > 0$, so $Y \cap I$ is bounded, which implies $d(Y \cap I) < \infty$. The set Y is the union of $Y \cap I$ and $Y - I$, so

$$d(Y) = \max\big(d(Y \cap I), d(Y - I)\big) = \max(d(Y \cap I), 0) = d(Y \cap I) < \infty.$$

(3) ⟹ (1): If S is a bounded subset of D, then $\sup S < \infty$, which implies $d(S) < \infty$. Therefore, R satisfies the bounded depth condition. □

Lemma 9.3. Let T be a solution of a semi-divide-and-conquer recurrence R with domain D and recursion set I, and suppose R has bounded depth of recursion on some subset S of D. Then $T|_S$ has a positive lower bound. If R is proper and the incremental cost is bounded on an initial subset of I containing $S \cap I$, then $T|_S = \Theta(1)$.

Proof. Let d be the depth-of-recursion function for R and let $n = d(S)$, so $n \in \boldsymbol{N}$. Define

$$E = \{x \in D : d(x) \leq n\},$$

so E contains S. Lemma 8.4 implies $T|_E$ has a positive lower bound, which is also a lower bound for $T|_S$.

Suppose R is proper and the incremental cost is bounded on some initial subset H of I containing $S \cap I$. Define $F = H \cup (D \backslash I)$, so F is an initial subset of D containing S. Lemma 8.6 combines with $H = F \cap I$ and $S \subseteq E \cap F$ to imply

$$\sup T|_S \leq \sup T|_{E \cap F} < \infty,$$

so $T|_S = \Theta(1)$. □

Corollary 9.4. Let R be a semi-divide-and-conquer recurrence that satisfies the bounded depth condition. Then R has a unique solution T, which has a positive lower bound on each bounded subset of its domain. Furthermore, T is locally $\Theta(1)$ if R is proper and the incremental cost of R is bounded on each bounded subset of the recursion set.

Proof. Satisfaction of the bounded depth condition implies R is finitely recursive. Corollary 8.5 (or Lemma 8.2) implies R has a unique solution T.

Let S be a bounded subset of the recurrence's domain, so $d(S) < \infty$ where d is the depth-of-recursion function. Lemma 9.3 implies $\inf T|_S > 0$.

Let g be the incremental cost of R, and let I be the recursion set. Suppose R is proper and g is bounded on each bounded subset of I. Let J be the minimum initial subset of I containing $S \cap I$. The set J is bounded because either $J = \emptyset$, or

$$\inf J = \inf I > 0$$

and

$$\sup J = \sup S < \infty.$$

Now g is bounded on J. Lemma 9.3 implies $T|_S = \Theta(1)$. Therefore T is locally $\Theta(1)$. □

The cheap, redundant variant below of Corollary 9.4 is more directly applicable for some purposes. For future convenience, we include some redundancy in the next couple propositions and elsewhere in this section.

Corollary 9.5. If R is a semi-divide-and-conquer recurrence that satisfies the bounded depth condition, then R has a unique solution T. Furthermore, T is locally $\Theta(1)$ if R is proper and the incremental cost of R has polynomial growth.

Proof. Corollary 9.4 (or Lemma 8.2 or Corollary 8.5) implies R has a unique solution T. Suppose R is proper, and the incremental cost has polynomial growth. Let I be the recursion set of R. If S is a bounded subset of I, then $\sup S < \infty$ and

$$\inf S \geq \inf I > 0.$$

If the incremental cost, g, has polynomial growth, then Corollary 2.23 implies g is bounded on each such S. The proposition follows from Corollary 9.4. □

Examples of finitely recursive divide-and-conquer recurrences that violate the bounded depth condition but have locally $\Theta(1)$ solutions. Let $t_0, t_1, t_2, \ldots$ be any increasing sequence of real numbers with $t_0 = 1$ and

$$\lim_{n \to \infty} t_n = 2.$$

Define a sequence $X_0, X_1, X_2, \ldots$ of disjoint half open intervals by $X_n = [t_n, t_{n+1})$, so that

$$[1, 2) = \bigcup_{n=0}^{\infty} X_n$$

and

$$[t_1, 2) = \bigcup_{n=1}^{\infty} X_n.$$

For all $n \in \boldsymbol{Z}^+$, define $\varphi_n: X_n \to X_{n-1}$ by $\varphi_n(x) = t_{n-1}$ for all $x \in X_n$. Define $\varphi: [t_1, 2) \to [1, 2)$ by $\varphi|_{X_n} = \varphi_n$ for all $n \in \boldsymbol{Z}^+$, and define $h: [t_1, \infty) \to \boldsymbol{R}$ by

$$h(x) = \begin{cases} \varphi(x) - x/2, & \text{for } x \in [t_1, 2) \\ 0, & \text{for } x \in [2, \infty), \end{cases}$$

so that

$$x/2 + h(x) = \begin{cases} \varphi(x), & \text{for } x \in [t_1, 2) \\ x/2, & \text{for } x \in [2, \infty). \end{cases}$$

Let $D = [1, \infty)$, $I = [t_1, \infty)$, $a = 1$, and $b = 1/2$. Define $f: X_0 \to \boldsymbol{R}$ by $f(x) = 1$, and let $g: I \to \boldsymbol{R}$ be identically zero. Then

$$(D, I, a, b, f, g, h)$$

is a divide-and-conquer recurrence. (Our definition in Section 20 of an *admissible* recurrence is also satisfied.)

For each non-negative integer n, we have $d(x) = n$ for all $x \in X_n$. Therefore, $d(x) < \infty$ for all $x \in [1, 2)$ where d is the depth-of-recursion function. We also conclude that $d(S) = \infty$ for $S = [1, 2)$, which implies the bounded depth condition is not satisfied.

If $x \geq 2$ and $m = \lfloor \log_2 x \rfloor$, we conclude from

$$2^m \leq x < 2^{m+1},$$

i.e.,

$$1 \leq \frac{x}{2^m} < 2,$$

that $x/2^m \in X_n$ for some non-negative integer n. Since $x/2^j \geq 2$ for each integer $j < m$, the depth of recursion at x is $m + n$, which is finite. Therefore, the recurrence is finitely recursive. By Corollary 8.5, the recurrence has a unique solution T, which satisfies

$$T(x) = \begin{cases} 1, & \text{for } x \in X_0 \\ T\left(\dfrac{x}{2} + h(x)\right), & \text{for } x \in D - X_0. \end{cases}$$

Induction on $d(x)$ shows that $T(x) = 1$ for all x in the domain D of the recurrence. In particular, T is locally $\Theta(1)$ although the bounded depth condition is not satisfied.

Suppose we modify the recurrence by letting $a = 1/2$ and defining $g: I \to \boldsymbol{R}$ to be the constant function $g(x) = 1/2$, i.e.,

$$T(x) = \begin{cases} 1, & \text{for } x \in X_0 \\ \frac{1}{2}T\left(\frac{x}{2} + h(x)\right) + \frac{1}{2}, & \text{for } x \in D - X_0. \end{cases}$$

The new divide-and-conquer recurrence has the same depth-of-recurrence function and also violates the bounded depth condition. Once again, the recurrence is finitely recursive with a unique solution T, which is locally $\Theta(1)$. Indeed, $T(x) = 1$ for all x in the domain D. (This recurrence is also admissible.)

Finite recursion of a divide-and-conquer recurrence with polynomial-growth incremental cost does not imply a locally $\boldsymbol{\Theta(1)}$ solution. Consider the (admissible) divide-and-conquer recurrence

$$T(x) = \begin{cases} 1, & \text{for } x \in X_0 \\ T\left(\frac{x}{2} + h(x)\right) + 1, & \text{for } x \in D - X_0 \end{cases}$$

where D, X_0, and h are defined as in the two previous examples. Lemma 2.3 implies the incremental cost has polynomial growth. This recurrence is also finitely recursive with a unique solution T. Unlike the aforementioned examples, $T(x) = d(x) + 1$ for all $x \in D$ where d is the depth-of-recursion function. Therefore T is unbounded on $[1, 2)$, which implies T is not locally $\Theta(1)$.

Definition. A semi-divide-and-conquer recurrence

$$(D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k)$$

satisfies the *ratio condition* if there exists a real number $\beta < 1$ such that

$$b_i x + h_i(x) \le \beta x$$

for all $x \in I$ and all $i \in \{1, \dots, k\}$. The recurrence satisfies the *strong ratio condition* if there exist $0 < \alpha \le \beta < 1$ such that

$$\alpha x \le b_i x + h_i(x) \le \beta x$$

for all $x \in I$ and all $i \in \{1, \dots, k\}$.

Of course, the ratio condition is equivalent to the existence of $\beta_1, \dots, \beta_k$ such that $\beta_i < 1$ and $b_i x + h_i(x) \le \beta_i x$ for all x in I and all $i \in \{1, \dots, k\}$: Given $\beta_1, \dots, \beta_k$, the ratio condition is satisfied with $\beta = \max\{\beta_1, \dots, \beta_k\}$. Conversely, if the ratio condition is satisfied, let $\beta_i = \beta$ for each i.

The strong ratio condition a priori implies the ratio condition. Similar to the ratio condition, the strong ratio condition is satisfied if and only if there exist $\alpha_1, \dots, \alpha_k$ and $\beta_1, \dots, \beta_k$ such that $0 < \alpha_i \le \beta_i < 1$ and $\alpha_i x \le b_i x + h_i(x) \le \beta_i x$ for all x in I and all $i \in \{1, \dots, k\}$: Given $\alpha_1, \dots, \alpha_k$ and $\beta_1, \dots, \beta_k$, the strong ratio condition is satisfied with $\alpha = \min\{\alpha_1, \dots, \alpha_k\}$ and $\beta = \max\{\beta_1, \dots, \beta_k\}$. Conversely, if the strong ratio condition is satisfied, let $\alpha_i = \alpha$ and $\beta_i = \beta$ for each i.

The ratio condition allows the uninteresting possibility that $\beta \le 0$, in which case the maximum depth of recursion is 1.

The ratio condition is equivalent to

$$\max_{1 \le i \le k} \left(\sup_{x \in I} \left(b_i + \frac{h_i(x)}{x} \right) \right) < 1.$$

The strong ratio condition is equivalent to the combination of the preceding inequality and

$$\min_{1 \le i \le k} \left(\inf_{x \in I} \left(b_i + \frac{h_i(x)}{x} \right) \right) > 0.$$

Lemma 9.6. If a semi-divide-and-conquer recurrence satisfies the ratio condition, then it is proper, satisfies the bounded depth condition, and has a unique solution T. Furthermore, T is locally $\Theta(1)$ if the incremental cost has polynomial growth.

Proof. Suppose

$$R = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k)$$

is a semi-divide-and-conquer recurrence that satisfies the ratio condition. Define real-valued functions $r_1, \dots, r_k$ on I by $r_i(x) = b_i x + h_i(x)$. By definition of the ratio condition, there exists $\beta < 1$ such that $r_i(x) \le \beta x$ for all x in I and all $i \in \{1, \dots, k\}$. Of course, $\beta x < x$, so R is proper.

By Corollary 9.5, we need only show that R satisies the bounded depth condition. Let S be any bounded subset of D. We will show R has bounded depth of recursion on S, i.e., $d(S) < \infty$ where d is the depth-of-recursion function.

If $\beta \le 0$, then $r_i(x) \le 0$ for all $x \in I$ and all $i \in \{1, \dots, k\}$. By definition of a semi-divide-and-conquer recurrence, we have $\inf I > 0$ and $r_i(x) \in D$ for all such x and i, so $r_i(x) \in D - I$, which implies $d(x) = 1$. By definition, I is non-empty, so $d(I) = 1$ and

$$d(S) \le d(D) = \max\bigl(d(I), d(D \backslash I)\bigr) = \max(1,0) = 1.$$

We may now assume $\beta > 0$. There exists $t \in (0, \inf I)$ because $\inf I > 0$. Define

$$E_j = D \cap (-\infty, t/\beta^j]$$

for each non-negative integer j. In particular,

$$E_0 = D \cap (-\infty, t] \subseteq D \backslash I,$$

so $d(E_0) = 0$. Let m be any non-negative integer for which $d(E_m) \leq m$. Observe that

$$r_i(x) \leq \beta x \leq t/\beta^m,$$

i.e., $r_i(x) \in E_m$, for all $x \in I \cap E_{m+1}$ and all $i \in \{1, \dots, k\}$, so

$$d(x) = 1 + \min_{1 \leq i \leq k} d\big(r_i(x)\big) \leq m + 1$$

for all such x. Therefore, $d(I \cap E_{m+1}) \leq m + 1$ and

$$d(E_{m+1}) = \max\big(d(E_{m+1} \backslash I), d(E_{m+1} \cap I)\big) \leq \max(0, m+1) = m + 1.$$

By induction, $d(E_n) \leq n$ for each non-negative integer n.

If S is a bounded subset of D, then there exists a non-negative integer δ such that $\sup S \leq t/\beta^\delta$, so $S \subseteq E_\delta$, which implies

$$d(S) \leq d(E_\delta) \leq \delta < \infty.$$

□

A divide-and-conquer recurrence that satisfies the bounded depth condition but violates the ratio condition. Let $D = [1, \infty)$ and $I = [2, \infty)$. Define increasing functions $\mu: \boldsymbol{Z}^+ \to [1/2\,, 1)$ and $\lambda: \boldsymbol{Z}^+ \to [2, 4)$ by

$$\mu(n) = \frac{n}{n+1}$$

and $\lambda(n) = 4\mu(n)$, so $\mu(1) = 1/2$, $\lambda(1) = 2$, and

$$\frac{\big(\lambda(n)\big)^2}{4} = \mu(n)\lambda(n) \in \big[1, \lambda(n)\big) \subset [1, 4).$$

Define $r: I \to D$ by

$$r(x) = \begin{cases} x^2/4\,, & \text{for } x \in \lambda(\boldsymbol{Z}^+) \\ x/2\,, & \text{for } x \notin \lambda(\boldsymbol{Z}^+). \end{cases}$$

Observe that $1 \leq r(x) < x$ for all $x \in I$; in particular, $r(x) \in D$. Define $h: I \to \boldsymbol{R}$ by

$$h(x) = r(x) - x/2,$$

i.e.,

$$r(x) = x/2 + h(x).$$

Then $h(x) = 0$ for all $x \in I \backslash \lambda(\boldsymbol{Z}^+)$. In particular, $h(x) = 0$ for all $x \geq 4$. Let $a \in \boldsymbol{R}^+$ and $c \in [0, \infty)$, so

$$T(x) = \begin{cases} 1, & \text{for } x \in D - I \\ aT\left(\dfrac{x}{2} + h(x)\right) + c, & \text{for } x \in I \end{cases}$$

is a divide-and-conquer recurrence with domain D and recursion set I. (Our definition in Section 20 of an *admissible* recurrence is also satisfied.) Let d be the depth-of-recursion function for the recurrence, so

$$d(x) = \begin{cases} 0, & \text{for } x \in D - I \\ d\big(r(x)\big) + 1, & \text{for } x \in I. \end{cases}$$

Observe that

$$r(t) \in [1, 2) = D - I$$

for all $t \in [2, 4) \backslash \lambda(\boldsymbol{Z}^+)$, so $d(t) = 1$ for all such t. Given $u \in \lambda(\boldsymbol{Z}^+)$, there exists $j \in \boldsymbol{Z}^+$ such that $u = \lambda(j)$ and

$$r(u) = \frac{4j^2}{(j+1)^2} \in [1, 4).$$

Observe that

$$j^2 + 1 \neq (j+1)^2,$$

so

$$r(u) \in [1, 4) \backslash \lambda(\boldsymbol{Z}^+) = [1, 2) \cup \big([2, 4) \backslash \lambda(\boldsymbol{Z}^+)\big).$$

Then $d\big(r(u)\big) \in \{0,1\}$, which implies $d(u) \leq 2$. If $u = 2$, i.e., $u = \lambda(1)$, then $r(u) = 1$, so $d(u) = 1$; if $u \neq 2$, then $u > 2$ and $\log_2 u > 1$. Therefore,

$$d(u) < 1 + \log_2 u$$

for all $u \in \lambda(\boldsymbol{Z}^+)$. Of course,

$$d(v) = 0 < 1 + \log_2 v$$

for all $v \in [1, 2)$. Suppose m is positive integer such that

$$d(w) < 1 + \log_2 w$$

for all $w \in [1, 2^m)$. Let

$$y \in [1, 2^{m+1}) \backslash [1, 2^m) = [2^m, 2^{m+1}),$$

so $y \in I$. If $y \in \lambda(\boldsymbol{Z}^+)$, then

$$d(y) < 1 + \log_2 y$$

as previously demonstrated. If $y \notin \lambda(\boldsymbol{Z}^+)$, then

$$r(y) = \frac{y}{2} \in [1, 2^m),$$

which implies

$$d(y) = 1 + d(y/2) < 2 + \log_2(y/2) = 1 + \log_2 y.$$

Therefore,

$$d(z) < 1 + \log_2 z$$

for all $z \in [1, 2^{m+1})$. By induction,

$$d(x) < 1 + \log_2 x$$

for all

$$x \in \bigcup_{n=1}^{\infty} [1, 2^n) = D.$$

Therefore, the recurrence satisfies the bounded depth condition. (In particular, the recurrence is finitely recursive.) Observe that

$$\sup_{n \in \mathbf{Z}^+} \frac{r(\lambda(n))}{\lambda(n)} = \sup_{n \in \mathbf{Z}^+} \frac{n}{n+1} = 1$$

and

$$\frac{r(x)}{x} = \frac{1}{2}$$

for all $x \in I - \lambda(\mathbf{Z}^+)$, so

$$\sup_{x \in I} \frac{r(x)}{x} = 1.$$

In particular, the ratio condition is violated.

The ratio and bounded depth conditions and the requirement that recursion sets have positive lower bounds. Lemma 9.6 is inapplicable to the recurrence

$$T(x) = \begin{cases} 1, & \text{for } x = 0 \\ T\left(\frac{x}{2}\right) + 1, & \text{for } x \in (0, \infty) \end{cases}$$

because our definition of a semi-divide-and-conquer recurrence is violated: the interval $(0, \infty)$ does not have a positive lower bound. The recurrence otherwise satisfies our definition of the ratio condition. However, the recurrence is infinitely recursive at each positive real number. Furthermore, there are infinitely many solutions and none of them are locally $\Theta(1)$; indeed, none of them are positive functions.

The simple observation below will prove useful in Section 20:

Lemma 9.7. Assume T is a locally $\Theta(1)$ solution of a semi-divide-and-conquer recurrence

$$R = (D, I, a_1, \ldots, a_k, b_1, \ldots, b_k, f, g, h_1, \ldots, h_k)$$

and J is a non-empty upper subset of I. Then

$$S = \big(D, J, a_1, \dots, a_k, b_1, \dots, b_k, T|_{D-J}, g|_J, h_1|_J, \dots, h_k|_J\big)$$

is also a semi-divide-and-conquer recurrence with T as a solution. If R satisfies one or more of the bounded depth, ratio, and strong ratio conditions, then S also satisfies each of those conditions satisfied by R. If R is proper, then S is proper.

Proof. The set J is an upper subset of D because J is an upper subset of the upper subset I of D. The recursion set I has a positive lower bound by definition of a semi-divide-and-conquer recurrence. Thus $\inf J \geq \inf I > 0$. Since J is non-empty, J satisfies the requirements of a recursion set. Since $\sup(D - J) \leq \inf J < \infty$, Lemma 9.1 implies $T|_{D-J}$ is $\Theta(1)$, i.e., $T|_{D-J}$ has a positive lower bound and finite upper bound. Satisfaction of the other requirements for a semi-divide-and-conquer recurrence with solution T is obviously inherited by S from R.

Suppose R satisfies the bounded depth condition, and X is a bounded subset of D. Let d_{D-I} and d_{D-J} be the depth of recursion functions for R relative to $D - I$ and $D - J$, respectively. The set $D - I$ is contained in $D - J$ because J is contained in I. Lemma 8.3 and satisfaction of the bounded depth condition by R imply

$$d_{D-J}(X) \leq d_{D-I}(X) < \infty.$$

Let d^* be the depth of recursion function for S. We conclude from $d^*(X) = d_{D-J}(X)$ that $d^*(X) < \infty$. Therefore S also satisfies the bounded depth condition.

The remaining assertions are (even more) trivial: If R satisfies the ratio condition, then $J \subseteq I$ implies S inherits satisfaction of the ratio condition. If S satisfies the strong ratio condition, then $J \subseteq I$ implies S inherits satisfaction of the strong ratio condition. Similarly, S is proper if R is proper. □

Lemma 9.8. Suppose

$$R = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k)$$

is a semi-divide-and-conquer recurrence with unbounded recursion set I such that

$$\lim_{x \to \infty} \frac{h_i(x)}{x} = 0$$

for all $i \in \{1, \dots, k\}$. There exists a non-empty upper subset J of I and real numbers α and β with $0 < \alpha < \beta < 1$ such that

$$\alpha x < b_i x + h_i(x) < \beta x$$

for all $x \in J$ and all $i \in \{1, \ldots, k\}$.

Proof. (Of course, the limit is taken over elements of I.) The unbounded set I is positive by definition, so indeed $\sup I = \infty$ as required by the limit. There exist $v, \ldots, v_k \in I$ such that

$$\left|\frac{h_i(x)}{x}\right| < \min\left(\frac{b_i}{2}, \frac{1 - b_i}{2}\right)$$

for all $x \in I \cap (v_i, \infty)$ and all $i \in \{1, \ldots, k\}$. Let $y = \max(v_1, \ldots, v_k)$, and define $J = I \cap (y, \infty)$, so that J is an upper subset of I. The set J is non-empty because $\sup I = \infty$. For all $i \in \{1, \ldots, k\}$, we have $J \subseteq I \cap (v_i, \infty)$, which implies

$$\left|\frac{h_i(x)}{x}\right| < \min\left(\frac{b_i}{2}, \frac{1 - b_i}{2}\right)$$

and

$$\frac{b_i}{2} = b_i - \frac{b_i}{2} < b_i - \left|\frac{h_i(x)}{x}\right| \leq b_i + \frac{h_i(x)}{x} \leq b_i + \left|\frac{h_i(x)}{x}\right| < b_i + \frac{1 - b_i}{2} = \frac{1 + b_i}{2}$$

for all $x \in J$. Let

$$\alpha = \min\left(\frac{b_1}{2}, \ldots, \frac{b_k}{2}\right) \quad \text{and} \quad \beta = \max\left(\frac{1 + b_1}{2}, \ldots, \frac{1 + b_k}{2}\right).$$

Then $0 < \alpha < \beta < 1$ and

$$\alpha x < b_i x + h_i(x) < \beta x$$

for all $x \in J$ and all $i \in \{1, \ldots, k\}$. □

Corollary 9.9. Suppose T is a locally $\Theta(1)$ solution of a semi-divide-and-conquer recurrence

$$R = (D, I, a_1, \ldots, a_k, b_1, \ldots, b_k, f, g, h_1, \ldots, h_k)$$

with unbounded recursion set I such that

$$\lim_{x \to \infty} \frac{h_i(x)}{x} = 0$$

for all $i \in \{1, \ldots, k\}$. There exists a non-empty upper subset J of I such that the semi-divide-and-conquer recurrence

$$S = \left(D, J, a_1, \ldots, a_k, b_1, \ldots, b_k, T|_{D-J}, g|_J, h_1|_J, \ldots, h_k|_J\right)$$

is proper and satisfies the bounded depth and strong ratio conditions. Furthermore, T is the unique solution of S.

Proof. (The unbounded set I is positive by definition, so $\sup I = \infty$ as required by the limit.) By Lemma 9.8, there exist a non-empty upper subset J of I and real numbers α and β with $0 < \alpha < \beta < 1$ such that

$$\alpha x < b_i x + h_i(x) < \beta x$$

for all $x \in J$ and all $i \in \{1, \dots, k\}$. Lemma 9.7 implies S is indeed a semi-divide-and-conquer recurrence with T as a solution. The inequalities above imply S satisfies the strong ratio condition. Lemma 9.6 implies S is proper, satisfies the bounded depth condition, and has a unique solution. □

Lemma 9.10. Let
$$R = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k)$$

be a semi-divide-and-conquer recurrence such that either the recursion set I is bounded or

$$\lim_{x \to \infty} \frac{h_i(x)}{x} = 0$$

for all $i \in \{1, \dots, k\}$. Let d be the depth-of-recursion function for R, and define

$$F_n = \{x \in D : d(x) \leq n\}$$

for each non-negative integer n. Then $\sup F_n < \infty$ for all such n. If the incremental cost, g, has polynomial growth, then

(1) If T is a solution of R, then T is $\Theta(1)$ on F_n for all $n \in \boldsymbol{N}$.

(2) If R satisfies the bounded depth condition, then R has a unique solution, which is locally $\Theta(1)$.

Proof. (By definition, $\inf I > 0$; if I is unbounded, then $\sup I = \infty$ as required by the limit.) We first show that $\sup F_n < \infty$ for all $n \in \boldsymbol{N}$. By definition of a semi-divide-and-conquer recurrence, the recursion set I is a non-empty upper subset of D, so

$$\sup F_n \leq \sup D = \sup I.$$

for all $n \in \boldsymbol{N}$. Thus we may assume $\sup I = \infty$. Lemma 9.8 implies there exists $\alpha \in (0,1)$ and a non-empty upper subset J of I such that

$$\alpha x \leq b_i x + h_i(x)$$

for all $x \in J$ and all $i \in \{1, \dots, k\}$. Of course, J is an upper subset of D because I is an upper subset of D. Furthermore, $\inf J > 0$ because $\inf I > 0$. Observe that $F_0 = D - I$, so

$$\sup F_0 \leq \inf I \leq \inf J = \frac{\inf J}{\alpha^0}.$$

Suppose $m \in \boldsymbol{N}$ such that

$$\sup F_m \leq \frac{\inf J}{\alpha^m}.$$

If $y \in F_{m+1} \cap J$, then $b_i y + h_i(y) \in F_m$ for all $i \in \{1, \ldots, k\}$, so

$$\alpha y \leq b_i y + h_i(y) \leq \frac{\inf J}{\alpha^m}.$$

Since J is an upper subset of D and $\inf J > 0$, we conclude that

$$\sup F_{m+1} = \max(\sup(F_{m+1} - J), \sup(F_{m+1} \cap J)) \leq \max\left(\inf J, \frac{\inf J}{\alpha^{m+1}}\right) = \frac{\inf J}{\alpha^{m+1}}.$$

By induction,

$$\sup F_n \leq \frac{\inf J}{\alpha^n} < \infty$$

for all $n \in \boldsymbol{N}$.

For the remainder of the proof, we assume g has polynomial growth. We now prove (1). If $F_0 = \emptyset$, then $F_n = \emptyset$ for all $n \in \boldsymbol{N}$, which implies T is $\Theta(1)$ on F_n for all such n. Therefore, we may assume $F_0 \neq \emptyset$. For all n, the set F_n contains F_0, so $F_n \neq \emptyset$.

Observe that $F_0 = D - I$, so T is $\Theta(1)$ on F_0 by definition of a semi-divide-and-conquer recurrence and its solutions. Let m be any natural number (including zero) for which T is $\Theta(1)$ on F_m. We conclude from $F_m \neq \emptyset$ that $\inf T(F_m)$ and $\sup T(F_m)$ are positive real numbers. If $F_{m+1} = F_m$, then T is $\Theta(1)$ on F_m. Now suppose $F_{m+1} \neq F_m$. The set F_{m+1} contains F_m, which contains F_0, i.e., $D - I$. Therefore, $F_{m+1} - F_m$ is a non-empty subset of I, the domain of g, so

$$\inf(F_{m+1} - F_m) \geq \inf I > 0.$$

We conclude from $\sup F_{m+1} < \infty$ that $\sup(F_{m+1} - F_m) < \infty$. Lemma 2.2(1) implies g is non-negative, so

$$\inf g(F_{m+1} - F_m) \geq 0.$$

Corollary 2.23 implies $\sup g(F_{m+1} - F_m) < \infty$. We have $b_i x + h_i(x) \in F_m$ for all $x \in F_{m+1} - F_m$ and all $i \in \{1, \ldots, k\}$, so

$$A \cdot \inf T(F_m) \leq T(x) \leq A \cdot \sup T(F_m) + \sup g(F_{m+1} - F_m)$$

for all such x where

$$A = \sum_{i=1}^{k} a_i.$$

Now

$$\inf T(F_{m+1}) = \min(\inf T(F_m), \inf T(F_{m+1} - F_m)) \geq \min(1, A) * \inf T(F_m) > 0$$

and

$$\sup T(F_{m+1}) = \max(\sup T(F_m), \sup T(F_{m+1} - F_m))$$

$$\leq \max(1, A) * \sup T(F_m) + \sup g(F_{m+1} - F_m) < \infty,$$

so T is $\Theta(1)$ on F_{m+1}. By induction, we obtain (1).

We now prove (2): Corollary 9.4 implies R has a unique solution T. Let S be any bounded subset of the domain D of R. Satisfaction of the bounded depth condition implies containment of S in F_n for some $n \in \boldsymbol{N}$, which combines with (1) to imply

$$\inf T(S) \geq \inf T(F_n) > 0$$

and

$$\sup T(S) \leq \sup T(F_n) < \infty,$$

i.e., $T|_S = \Theta(1)$. Therefore, T is locally $\Theta(1)$. □

Noise constraint can be loosened. The condition

$$\lim_{x \to \infty} \frac{h_i(x)}{x} = 0$$

of propositions 9.8, 9.9, and 9.10 can be replaced by the combination of $L > 0$ and $U < 1$ where

$$L = \min_{1 \leq i \leq k} \left(\liminf_{x \to \infty} \left(b_i + \frac{h_i(x)}{x} \right) \right)$$

and

$$U = \max_{1 \leq i \leq k} \left(\limsup_{x \to \infty} \left(b_i + \frac{h_i(x)}{x} \right) \right).$$

The conclusion of Lemma 9.8 is satisfied by all $\alpha \in (0, L)$ and $\beta \in (U, 1)$. However, we have no need for this refinement.

10. Akra-Bazzi Integrals

All of the propositions in [Le] involve Akra-Bazzi integrals of the form

$$\int_a^x u^c g(u)\, du$$

where $a > 0$ and c are real numbers, and g plays a role similar to the incremental cost of a semi-divide-and-conquer recurrence. ($c = -p - 1$ where p is the Akra-Bazzi exponent.) However, there is no explicit integrability requirement for g in [Le]. Our replacements for Leighton's propositions include explicit integrability conditions. Although [Le] mentions the derivative of g, the function g need not be differentiable or even continuous.

Definition. A *tame* function is a polynomial-growth, locally Riemann integrable, real-valued function on a non-empty, positive interval.

The following three propositions list some obvious consequences of the definition:

Lemma 10.1.

(1) Tame functions are either positive or identically zero.

(2) The restriction of a tame function to a non-empty subinterval of its domain is also tame.

Proof. Lemma 2.7 implies (1). Local Riemann integrability is obviously inherited by restrictions to non-empty subintervals. The proposition follows from Lemma 2.2(2). □

Lemma 10.2. Let *I* be a non-empty, positive interval. Sums and products of tame functions on *I* are tame. Non-negative, constant functions on *I* are tame as are non-negative scalar multiples of tame functions. Reciprocals of positive tame functions are tame as are quotients of tame functions on *I* with positive denominators.

Proof. The specified functions are locally Riemann integrable (standard facts easily proved, e.g., by Lebesgue's criterion for Riemann integrability). Polynomial growth follows from Lemma 2.3 and Corollaries 2.15, 4.3, and 4.4. □

Akra-Bazzi integrands are the specific case of interest:

Corollary 10.3. If $g: I \to \boldsymbol{R}$ is a tame function on a non-empty, positive interval I, and c is a real number, then the function $f: I \to \boldsymbol{R}$ defined by $f(x) = x^c g(x)$ is also tame.

Proof. The function $h: I \to \boldsymbol{R}$ defined by $h(x) = x^c$ has polynomial growth by Lemma 4.1(2) and is locally Riemann integrable, so h is tame. Lemma 10.2 implies f is tame. □

Sets of Measure Zero. A set S of real numbers is defined to have measure zero if for each $\varepsilon > 0$, there exists a countable set C of open intervals such that

$$S \subseteq \bigcup_{A \in C} A,$$

i.e., C is a cover of S, and

$$\sum_{A \in C} length(A) < \varepsilon.$$

An equivalent definition of *measure zero* is obtained by replacing *open intervals* with *non-empty, bounded open intervals*.

The definition above of *measure zero* is equivalent to the definition of a *Lebesgue measurable* real set with *Lebesgue measure* zero as defined in [Ta] and elsewhere. However, we do not require any knowledge of Lebesgue measure.

The empty set and real singletons have measure zero. Countable unions of sets of measure zero also have measure zero. In particular, countable real sets have measure zero. Subsets of sets of measure zero also have measure zero.

We claim that no non-degenerate compact interval I has measure zero: Let S be any countable cover of I whose elements are non-empty, bounded open intervals. An inductive argument (See Lemma 5.1.1 of [Ed]) shows that

$$length(I) < \sum_{X \in S} length(X).$$

Every non-degenerate interval contains a non-degenerate compact subinterval, which does not have measure zero. Therefore, non-degenerate intervals do not have measure zero, i.e., an interval has measure zero if and only if it is degenerate.

Lebesgue's criterion for Riemann Integrability: A real-valued function f on a non-empty, compact interval is Riemann integrable if and only if f is bounded and the set of discontinuities of f have measure zero ([Ap], Definition 7.43 and Theorem 7.48). The reference contains a slightly different version of the criterion, which is equivalent to the formulation here because (1) all Riemann integrable functions on compact intervals are bounded, and (2) our convention of Riemann integrability for all real-valued functions on real singletons is compatible with inclusion of such functions in Lebesgue's criterion ([Ap] does not define Riemann integrability for functions on such domains): Every such function is bounded and continuous; continuity implies the set of discontinuities is the empty set, which has measure zero.

We use Lebesgue's criterion in several places. (Do not be misled by our mention of Lebesgue. No knowledge of Lebesgue integration is required to understand this document.) Of course, Riemann or Darboux sums can be used to provide simple alternative proofs wherever we apply Lebesgue's criterion.

We remind ourselves of an elementary fact:

Lemma 10.4. If f is a positive, Riemann integrable function on $[a, b]$ where a and b are real numbers with $a < b$, then

$$\int_a^b f(x)dx > 0.$$

Proof. Let S be the set of discontinuities of f, so S has measure zero by Lebesgue's criterion for Riemann integrability. Then S is a proper subset of $[a, b]$ because $a < b$. Therefore, f is continuous at some $t \in [a, b]$. Since $a < b$, there exists a closed interval $[y, z] \subseteq [a, b]$ of positive length such that $t \in [y, z]$ and $f(x) > f(t)/2$ for all $x \in [y, z]$. Since f is non-negative, we know that

$$\int_a^y f(x)dx \geq 0 \text{ and } \int_z^b f(x)dx \geq 0.$$

Therefore,

$$\int_a^b f(x)dx \geq \int_y^z f(x)dx > \frac{f(t)(z-y)}{2} > 0.$$

□

Our replacements for the propositions of [Le] involve integrals that may be improper. We now examine convergence of improper integrals with tame integrands.

Lemma 10.5. Let I be a non-empty, positive interval with $x_0 \notin I$ and $x_0 > 0$ where $x_0 = \inf I$. Define $I^* = I \cup \{x_0\}$. Suppose $f: I \to \boldsymbol{R}$ is tame and $f^*: I^* \to \boldsymbol{R}$ is an extension of f, i.e., $f^*|_I = f$. Then f^* is locally Riemann integrable, and the improper integral

$$\int_{x_0}^{x} f(u)du = \lim_{t \to x_0^+} \int_t^x f(u)du$$

converges to the real value

$$\int_{x_0}^{x} f^*(u)du$$

for all $x \in I$. If f is a positive function, then f^* is tame if and only f^* is positive, i.e., $f^*(x_0) > 0$. If f is not positive, i.e., f is identically zero, then f^* is tame if and only f^* is identically zero, i.e., $f^*(x_0) = 0$. (In particular, there exists a tame extension of f to I^* regardless of whether f is positive). If f is positive, the improper integral is positive.

Proof. We claim f^* is bounded on each bounded subset B of I^*: Corollary 2.23 and $x_0 > 0$ imply f is bounded on $B \cap I$. The function f^* agrees with f on $B \cap I$ and is therefore also bounded on $B \cap I$. Of course, f^* is bounded on the set $B \cap \{x_0\}$, which has at most one element. Therefore, f^* is bounded on $B = (B \cap I) \cup (B \cap \{x_0\})$ as claimed.

The function f^* is Riemann integrable on every non-empty compact subinterval of I because f is locally Riemann integrable and f^* agrees with f on I. The function f^* is also Riemann integrable on $[x_0, x_0]$, i.e., $\{x_0\}$, according to our convention that every real valued function on a real singleton is Riemann integrable.

We will show that f^* is Riemann integrable on every non-empty compact subinterval K of I^*. We may assume $K \nsubseteq I$ and $K \neq \{x_0\}$, so $K = [x_0, c]$ for some $c \in I$.

Let $\varepsilon > 0$ and $b \in (x_0, c) \cap (x_0, x_0 + \varepsilon/4)$. Then $\emptyset \neq [b, c] \subseteq I$, which implies f^* is Riemann integrable on $[b, c]$. Let Y be the set of points in $[b, c]$ at which the restriction of f^* to $[b, c]$ is discontinuous. Lebesgue's criterion for Riemann integrability implies Y has measure zero. Let Z be the set of points in K at which the restriction of f^* to K is discontinuous. Then $Z \cap (b, c] \subseteq Y$, which implies $Z \cap (b, c]$ has measure zero. There exists a countable set R of open intervals such that the set

$$R^* = \bigcup_{A \in R} A$$

contains $Z \cap (b, c]$, and

$$\sum_{A \in R} length(A) < \frac{\varepsilon}{2}.$$

Define the open interval

$$H = \left(x_0 - \frac{\varepsilon}{4}, x_0 + \frac{\varepsilon}{4}\right),$$

so $[x_0, b] \subset H$ and $length(H) = \varepsilon/2$. Define $S = R \cup \{H\}$, so S is a countable set of open intervals. Let

$$S^* = \bigcup_{A \in S} A,$$

i.e., $S^* = R^* \cup H$. Then

$$Z = (Z \cap [x_0, b]) \cup (Z \cap (b, c]) \subseteq H \cup R^* = S^*.$$

Furthermore,

$$\sum_{A \in S} length(A) = length(H) + \sum_{A \in R} length(A) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon\,.$$

Therefore, Z has measure zero. Boundedness of f^* on K and Lebesgue's criterion for Riemann integrability imply f^* is Riemann integrable on K. We conclude that f^* is locally Riemann integrable.

We now verify the claimed convergence of the improper integral. Let $x \in I$, so $x > x_0$. Define

$$M = \sup_{r \in [x_0, x]} |f^*(r)|,$$

so $M < \infty$. For all $\delta > 0$, we have

$$\left| \int_{x_0}^{x} f^*(u)du - \int_{t}^{x} f(u)du \right| = \left| \int_{x_0}^{t} f^*(u)du \right| \leq M \cdot (t - x_0) < \delta$$

when

$$x_0 < t < \min\left(x, x_0 + \frac{\delta}{M}\right).$$

Therefore,

$$\lim_{t \to x_0^+} \int_{t}^{x} f(u)du = \int_{x_0}^{x} f^*(u)du \in \boldsymbol{R}.$$

Since f^* is locally Riemann integrable, f^* is tame if and only if f^* has polynomial growth. Lemma 10.1(1) implies either f is a positive function or f is identically zero.

If f is positive, then Lemma 2.7, Corollary 2.26, and non-emptiness of I imply f^* is tame if and only if $f^*(x_0) > 0$. If f is identically zero, then Lemmas 2.3 and 2.7 combine with non-emptiness of I to imply f^* is tame if and only $f^*(x_0) = 0$.

We now exhibit a tame function $g: I^* \to \boldsymbol{R}$ with $g|_I = f$. Let $g(x_0) = 1$ if f is positive; otherwise let $g(x_0) = 0$. If f is positive, then g is positive and Lemma 10.4 implies

$$\int_{x_0}^{x} g(u)du > 0,$$

i.e.,

$$\int_{x_0}^{x} f(u)du > 0.$$

□

Divergent integral with $\boldsymbol{inf\ I = 0}$. The function $u \mapsto 1/u$ on $(0, \infty)$ is tame, and

$$\lim_{t \to 0^+} \int_t^x \frac{1}{u} du = \infty$$

for all $x > 0$.

Under appropriate conditions, the Akra-Bazzi formula is locally $\Theta(1)$:

Lemma 10.6. Suppose g is a tame function on a non-empty, positive interval I, and $c \in I \cup (\{\inf I\} - \{0\})$. Let p be a real number. The function $A: I \cap [c, \infty) \to \boldsymbol{R}$ defined by

$$A(x) = x^p \left(1 + \int_c^x \frac{g(u)}{u^{p+1}} du\right)$$

is locally $\Theta(1)$. The integral converges if it is improper, i.e., if $c \notin I$.

Proof. Corollary 10.3 implies the function $f: I \to \boldsymbol{R}$ defined by $f(u) = g(u)/u^{p+1}$ is tame. Observe that $c > 0$. Lemma 10.5 implies the integral converges if it is improper, i.e. $c = \inf I \notin I$.

Let S be a bounded subset of $I \cap [c, \infty)$. We claim A is $\Theta(1)$ on S. By our definition, the empty function is $\Theta(1)$. Therefore, we may assume $S \neq \phi$. Let $M = \sup S$, so $c \leq M < \infty$ and $S \subseteq I \cap [c, M]$. Define

$$W = \sup f(I \cap [c, M]).$$

Corollary 2.23 implies $W < \infty$. Lemma 10.1(1) implies f is non-negative, so $W \geq 0$ and

$$0 \leq \int_c^x f(u)\, du \leq (x - c)W \leq (M - c)W < MW$$

for all $x \in S$. The function x^p is monotonic on $(0, \infty)$, so

$$0 < \min(c^p, M^p) \leq A(x) \leq (\max(c^p, M^p)) \cdot (1 + MW) < \infty$$

for all $x \in S$. In particular, $A|_S = \Theta(1)$. Therefore, A is locally $\Theta(1)$. □

We turn our attention to the lower limit of integration and its effect on the Akra-Bazzi formula.

Lemma 10.7. Suppose f is a positive, tame function on a non-empty, positive interval I. Define $J = I \cup (\{\inf I\} - \{0\})$. If $s, t \in J$ and $y \in \boldsymbol{R}$ such that $y > \max\{s, t\}$, then there exist positive real numbers λ_1 and λ_2 such that

$$\lambda_1 \int_s^x f(u)\,du \le \int_t^x f(u)\,du \le \lambda_2 \int_s^x f(u)\,du$$

for all $x \in I \cap [y, \infty)$.

Proof. Let $a \in J$. If $a \notin I$, then $a = \inf I > 0$, and Lemma 10.5 implies the improper integral

$$\int_a^z f(u)\,du$$

converges for all $z \in I$. In particular, any improper integral that appears in the statement of the current proposition is convergent (i.e., if $\{s, t\} \nsubseteq I$).

The inequality $y > s$ implies $y > \inf I$. If $y \notin I$, then $y \ge \sup I$ and

$$I \cap [y, \infty) = (I \cap \{y\}) \cup \big(I \cap (y, \infty)\big) \subseteq I \cap (\sup I\,, \infty) = \emptyset,$$

so the lemma is vacuously satisfied with $\lambda_1 = \lambda_2 = 1$. Therefore, we may assume $y \in I$. We may also assume $s \neq t$; otherwise the lemma is again satisfied with $\lambda_1 = \lambda_2 = 1$.

If f is not a positive function, then Lemma 10.1(1) implies f is identically zero, so

$$\int_s^x f(u)\,du = \int_t^x f(u)\,du = 0$$

for all $x \in I \cap [y, \infty)$ and the lemma is satisfied by $\lambda_1 = \lambda_2 = 1$. Therefore, we may assume f is positive. Lemmas 10.4 and 10.5 imply

$$\int_\alpha^\beta f(u)\,du > 0$$

for all $\alpha, \beta \in J$ that satisfy $\alpha < \beta$.

Let $c = \min(s, t)$ and $d = \max(s, t)$, so $c, d, y \in J$ and $0 < c < d < y$. Define

$$A = \int_c^d f(u)\,du \quad \text{and} \quad B = \int_d^y f(u)\,du,$$

so $A > 0$ and $B > 0$. Let $k = B/(A + B)$, so $k > 0$. Let $x \in I \cap [y, \infty)$, so

$$\int_y^x f(u)\,du \ge 0.$$

(The integral above is zero if and only if $x = y$.) Observe that

$$\int_d^x f(u)\,du = B + \int_y^x f(u)\,du \geq B.$$

Therefore,

$$\frac{\int_c^x f(u)\,du}{\int_d^x f(u)\,du} = \frac{A + \int_d^x f(u)\,du}{\int_d^x f(u)\,du} \leq \frac{A}{B} + 1 = \frac{1}{k}.$$

The denominators above are positive. Now

$$k\int_c^x f(u)\,du \leq \int_d^x f(u)\,du < A + \int_d^x f(u)\,du = \int_c^x f(u)\,du.$$

If $s < t$, then $c = s$ and $d = t$; the proposition is satisfied with $\lambda_1 = k$ and $\lambda_2 = 1$. If $s > t$, then $c = t$ and $d = s$; the proposition is satisfied with $\lambda_1 = 1$ and $\lambda_2 = 1/k$. □

Dangerous Bend. The condition $x \in I \cap [y, \infty)$ of Lemma 10.7 cannot be weakened to $x \in I \cap (d, \infty)$ when f is positive, $s \neq t$, and $d = \max(s,t) \neq \sup I$. Let $c = \min(s,t)$, so $c < d$ and

$$\int_c^d f(u)\,du > 0.$$

Observe that

$$\lim_{x \to d^+} \int_d^x f(u)\,du = 0.$$

For all $k > 0$ there exists $x \in I \cap (d, \infty)$ such that

$$\int_d^x f(u)\,du < k\int_c^d f(u)\,du < k\int_c^x f(u)\,du.$$

If $s < t$, then $c = s$ and $d = t$; there is no $\lambda_1 > 0$ that satisfies

$$\lambda_1 \int_s^x f(u)\,du \leq \int_t^x f(u)\,du$$

for all $x \in I \cap (d, \infty)$. If $s > t$, then $d = s$ and $c = t$; there is no $\lambda_2 > 0$ that satisfies

$$\int_t^x f(u)\,du \leq \lambda_2 \int_s^x f(u)\,du$$

for all such x.

Lemma 10.8. Suppose g is a tame function on a non-empty, positive interval I. Define $J = I \cup (\{\inf I\} - \{0\})$. Let $s, t \in J$, and let p be a real number. Define the functions

$$A: I \cap [s, \infty) \to \boldsymbol{R}^+$$

and

$$B: I \cap [t, \infty) \to \boldsymbol{R}^+$$

by

$$A(x) = x^p \left(1 + \int_s^x \frac{g(u)}{u^{p+1}} du\right)$$

and

$$B(x) = x^p \left(1 + \int_t^x \frac{g(u)}{u^{p+1}} du\right).$$

There exist positive real numbers λ_1 and λ_2 such that

$$\lambda_1 A(x) \le B(x) \le \lambda_2 A(x)$$

for all $x \in I \cap [m, \infty)$ where $m = \max(s, t)$. If $s \notin I$ or $t \notin I$, the improper integral in the definition of $A(x)$ or $B(x)$, respectively, is convergent for all x in the corresponding domain.

Proof. Corollary 10.3 implies the function $f: I \to \boldsymbol{R}$ defined by $f(u) = g(u)/u^{p+1}$ is tame. If $\alpha \in J - I$, then $\alpha = \inf I > 0$, and Lemma 10.5 implies the improper integral

$$\int_\alpha^z f(u)\, du$$

converges for all $z \in I$. In particular, any improper integral that appears in the statement of the current proposition is convergent (i.e., if $\{s, t\} \nsubseteq I$). The function f is non-negative by Lemma 10.1(1), so

$$\int_a^b f(u)\, du \ge 0$$

whenever $a \in J$ and $b \in I$ such that $a \le b$. Therefore, A and B are positive, real-valued functions as claimed.

If f is not positive, then Lemma 10.1(1) implies f is identically zero, so

$$A(x) = B(x) = x^p$$

for all $x \in I \cap [m, \infty)$, and the proposition is satisfied with $\lambda_1 = \lambda_2 = 1$. We now assume f is a positive function.

Let $c = \min\{s, t\}$ and $y = m + 1$. Lemma 10.7 implies there exist positive real numbers k_1 and k_2 such that

$$k_1 \int_s^x f(u)\, du \le \int_t^x f(u)\, du \le k_2 \int_s^x f(u)\, du$$

for all $x \in I \cap [y, \infty)$. Let $c_1 = \min(1, k_1)$ and $c_2 = \max(1, k_2)$, so $c_1, c_2 \in \boldsymbol{R}^+$ and

$$c_1 A(x) \le B(x) \le c_2 A(x)$$

for all such x. Lemma 10.6 implies there exist positive real numbers α_1, α_2, β_1, and β_2 such that $\alpha_1 \le A(x) \le \alpha_2$ and $\beta_1 \le B(x) \le \beta_2$ for all $x \in I \cap [m, y)$. For all such x, we have

$$\frac{\beta_1}{\alpha_2} A(x) \le B(x) \le \frac{\beta_2}{\alpha_1} A(x).$$

The proposition holds with $\lambda_1 = \min\{c_1, \beta_1/\alpha_2\}$ and $\lambda_2 = \max\{c_2, \beta_2/\alpha_1\}$ because

$$I \cap [m, \infty) = \big(I \cap [m, y)\big) \cup \big(I \cap [y, \infty)\big).$$

□

In Section 20, we show under mild assumptions that validity of a strong form of the modified Akra-Bazzi formula is essentially independent of the lower limit of integration. The following proposition plays a critical role in the proof.

Lemma 10.9. Let D be a set of real numbers, and let I and J be non-empty, upper subsets of D with $s = \inf I > 0$ and $t = \inf J > 0$. Suppose g is a tame function whose domain contains $I \cup J$. Let p be a real number, and let $A, B : D \to \boldsymbol{R}$ be real-valued functions on D satisfying

$$A(x) = \begin{cases} x^p \left(1 + \int_s^x \frac{g(u)}{u^{p+1}} du\right), & \text{for } x \in I \\ \Theta(1), & \text{for } x \in D - I \end{cases}$$

and

$$B(x) = \begin{cases} x^p \left(1 + \int_t^x \frac{g(u)}{u^{p+1}} du\right), & \text{for } x \in J \\ \Theta(1), & \text{for } x \in D - J. \end{cases}$$

If $s \notin domain(g)$ or $t \notin domain(g)$, the corresponding improper integral is convergent for all $x \in I$ or all $x \in J$, respectively. There exist positive real numbers λ_1 and λ_2 such that

$$\lambda_1 A(x) \le B(x) \le \lambda_2 A(x)$$

for all $x \in D$.

Proof. Let H be the domain of g. By definition of a tame function, H is a non-empty, positive interval. Define $K = H \cup (\{\inf H\} - \{0\})$.

We claim $s \in K$: The set H contains I, so $s \ge \inf H$. If $s = \inf H$, then $\inf H > 0$, so $\inf H \in K$, i.e., $s \in K$ as claimed. Suppose instead that $s > inf\ H$, so there exists $h \in H$ such that $h < s$. Non-emptiness of I implies there exists $z \in I$, so $s \le z$. Containment of I in H implies $z \in H$. Connectivity of H and the chain of inequalities $h < s \le z$ imply $s \in H$. We conclude from $H \subseteq K$ that $s \in K$. Similarly, $t \in K$.

Corollary 10.3 implies the function $u \mapsto g(u)/u^{p+1}$ on H is tame. Let $c = \min(s,t)$, so $c \in K$ and $c > 0$. If $\{s,t\} \nsubseteq H$, then $c = \inf H \notin H$, and Lemma 10.5 implies the improper integral

$$\int_c^x \frac{g(u)}{u^{p+1}} du$$

converges for all $x \in H$.

Define functions $A^*: H \cap [s,\infty) \to \boldsymbol{R}$ and $B^*: H \cap [t,\infty) \to \boldsymbol{R}$ by

$$A^*(x) = x^p \left(1 + \int_s^x \frac{g(u)}{u^{p+1}} du\right)$$

and

$$B^*(x) = x^p \left(1 + \int_t^x \frac{g(u)}{u^{p+1}} du\right).$$

Observe that $A|_I = A^*|_I$ and $B|_J = B^*|_J$.

Lemma 10.8 implies there exist positive real numbers α_1 and α_2 such that

$$\alpha_1 A^*(x) \leq B^*(x) \leq \alpha_2 A^*(x)$$

for all $x \in H \cap [m,\infty)$ where $m = \max(s,t)$. Observe that

$$D \cap (m,\infty) \subseteq I \cap J \subseteq \big(H \cap [s,\infty)\big) \cap \big(H \cap [t,\infty)\big) = H \cap [m,\infty),$$

so

$$\alpha_1 A(x) \leq B(x) \leq \alpha_2 A(x)$$

for all $x \in D \cap (m,\infty)$. Observe that $I \cap (s,m] = D \cap (s,m]$. Lemma 10.6 implies A^* is $\Theta(1)$ on $I \cap [s,m]$, i.e., A is $\Theta(1)$ on

$$I \cap [s,m] = (I \cap \{s\}) \cup (D \cap (s,m]).$$

By hypothesis, A is $\Theta(1)$ on

$$D - I = \big(D \cap (-\infty,s)\big) \cup \big((D - I) \cap \{s\}\big),$$

so A is $\Theta(1)$ on

$$(D - I) \cup (I \cap [s,m]) = D \cap (-\infty,m].$$

Similarly, B is $\Theta(1)$ on $D \cap (-\infty,m]$. There exist positive real numbers γ_1, γ_2, δ_1, and δ_2 such that

$$\gamma_1 \leq A(x) \leq \gamma_{2,}$$

$$\delta_1 \leq B(x) \leq \delta_2,$$

and

$$\frac{\delta_1}{\gamma_{2,}}A(x) \leq B(x) \leq \frac{\delta_2}{\gamma_1}A(x)$$

for all $x \in D \cap (-\infty, m]$. Let

$$\lambda_1 = \min\left(\alpha_1, \frac{\delta_1}{\gamma_{2,}}\right) \text{ and } \lambda_2 = \max\left(\alpha_2, \frac{\delta_2}{\gamma_1}\right),$$

so $\lambda_1, \lambda_2 \in \boldsymbol{R}^+$ and

$$\lambda_1 A(x) \leq B(x) \leq \lambda_2 A(x)$$

for all $x \in D$. □

11. Replacement for Leighton's Theorem 1

We offer an extremely modest revision of Theorem 1 of [Le] that incorporates the modifications discussed in Section 7, includes an explicit integrability requirement, and has a change in the lower limit of integration to accommodate the new domain of the function g.

Leighton's Theorem 1 (revised). Suppose $a_1, \dots, a_k \in \boldsymbol{R}^+$ and $b_1, \dots, b_k \in (0,1)$ for some $k \in \boldsymbol{Z}^+$. Let $x_0 \in \boldsymbol{R}$ such that $x_0 \geq 1/b_i$ for all $i \in \{1, \dots, k\}$. If $f: [1, x_0] \to \boldsymbol{R}^+$ is $\Theta(1)$, and g is a tame function on (x_0, ∞), then there exists exactly one function $T: [1, \infty) \to \boldsymbol{R}$ such that $T|_{[1,x_0]} = f$ and

$$T(x) = \sum_{i=1}^{k} a_i T(b_i x) + g(x)$$

for all $x > x_0$. Furthermore,

$$T(x) = \Theta\left(x^p \left(1 + \int_{x_0}^{x} \frac{g(u)}{u^{p+1}} du\right)\right)$$

where p is the unique real number for which

$$\sum_{i=1}^{k} a_i b_i^p = 1.$$

Of course,

$$(D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k)$$

is a divide-and-conquer recurrence where $D = [1, \infty)$, $I = (x_0, \infty)$, and the functions $h_1, \dots, h_k: I \to \boldsymbol{R}$ are identically zero. The recurrence satisfies the ratio condition with

$$\beta = \max_{1 \leq i \leq k} b_i$$

and has a unique solution, T, by Lemma 9.6. The existence and uniqueness of p will be established later as the entirely straightforward Lemma 11.1. The Akra-Bazzi integrand, $g(u)/u^{p+1}$, is a tame function on I by Corollary 10.3. The modified Akra-Bazzi integral in our revision of Leighton's Theorem 1 is improper:

$$\int_{x_0}^{x} \frac{g(u)}{u^{p+1}} du = \lim_{t \to x_0^+} \int_{t}^{x} \frac{g(u)}{u^{p+1}} du,$$

which converges for all $x > x_0$ by Lemma 10.5. The Akra-Bazzi integral is undefined for $x \leq x_0$ because the domain of g is (x_0, ∞).

Some very minor complications, such as an improper Akra-Bazzi integral, could be avoided by specifying $[x_0, \infty)$ as the domain of g in our replacement for Leighton's Theorem 1. By Lemma 10.5, g has a tame extension $g^*: [x_0, \infty) \to \boldsymbol{R}$. Corollary 10.3 implies the function $g^*(u)/u^{p+1}$ on $[x_0, \infty)$ is tame for all such g^*. Lemma 10.5 implies

$$\lim_{t \to x_0^+} \int_{t}^{x} \frac{g(u)}{u^{p+1}} du, = \int_{x_0}^{x} \frac{g^*(u)}{u^{p+1}} du$$

for all $x \in (x_0, \infty)$ and all such g^*. Furthermore, Lemma 10.1(2) implies the restriction of any tame function from $[x_0, \infty)$ to (x_0, ∞) is also tame. Thus the choice between (x_0, ∞) and $[x_0, \infty)$ as the domain of g is only a matter of taste and convenience.

Our revision of Theorem 1 relies on a corresponding revision of Lemma 1 of [Le]:

Leighton's Lemma 1 (revised). Suppose g is a tame function on a non-empty positive interval I. If $p \in \boldsymbol{R}$ and $b_1, \ldots, b_k \in (0,1)$ for some $k \in \boldsymbol{Z}^+$, then there exist $\lambda_1, \lambda_2 \in \boldsymbol{R}^+$ such that

$$\lambda_1 g(x) \leq x^p \int_{b_i x}^{x} \frac{g(u)}{u^{p+1}} du \leq \lambda_2 g(x)$$

for all $i \in \{1, \ldots, k\}$ and all $x \in I$ for which $b_1 x, \ldots, b_k x \in I$.

Of course, the revised Lemma 1 is vacuous if the interval I is too small. The proposition can be proved in the same fashion as the original Lemma 1 of [Le]. The lemma is applied with $I = (x_0, \infty)$ in the proof of the new Theorem 1. In effect, the condition $x \geq 1$ of Leighton's Lemma 1 is changed to $b_i x > x_0$ for all $i \in \{1, \ldots, k\}$. We also substitute our definition of polynomial growth and include an integrability condition.

When $\inf I > 0$ (as in the application to Theorem 1), Lemma 10.5 can be used to replace the condition $b_1 x, \ldots, b_k x \in I$ in the revised Lemma 1 with $b_1 x, \ldots, b_k x \in I \cup \{\inf I\}$, i.e., $b_i x > \inf I$ can be replaced with $b_i x \geq \inf I$. The resulting integral is improper for some choice of b_i and x. We shall not need this further refinement of Lemma 1.

The proof of Theorem 1 in [Le] uses a partition of $[1, \infty)$ into an infinite sequence of intervals, and proceeds by induction on the interval index. We use a partition of (x_0, ∞) instead. Let

$$b = \min_{1 \le i \le k} (\min(b_i, 1 - b_i)).$$

In particular, $0 < b < 1$. Define $x_1 = x_0/b$, $I_0 = (x_0, x_1]$, and $I_j = (x_1 + j - 1, x_1 + j]$ for each positive integer j. The intervals $I_0, I_1, \dots$ are disjoint, and their union is (x_0, ∞).

Lemmas 9.6 and 10.6 imply $T(x)$ and

$$x^p \left(1 + \int_{x_0}^{x} \frac{g(u)}{u^{p+1}} du\right)$$

are $\Theta(1)$ on I_0 as required by the base case of the induction.

We argue as in [Le]. Suppose $x \in I_j$ for some positive integer j, so

$$b_i x > b_i(x_1 + j - 1) \ge b_i x_1 \ge b x_1 = x_0$$

and

$$b_i x \le b_i(x_1 + j) < b_i x_1 + j = x_1 + j - (1 - b_i)x_1 \le x_1 + j - b x_1 = x_1 + j - x_0$$

$$< \ x_1 + j - 1$$

for all $i \in \{1, \dots, k\}$. Therefore,

$$b_i x \in (x_0, x_1 + j - 1]) = \bigcup_{n=0}^{j-1} I_n \, ,$$

which implies $b_i x \in I_n$ for some $0 \le n < j$ as required by the inductive step of the proof of Theorem 1. Since $b_i x > x_0$, the revised Lemma 1 is applicable with $I = (x_0, \infty)$ to the integral

$$\int_{b_i x}^{x} \frac{g(u)}{u^{p+1}} du,$$

which appears in the inductive step in the proof of Theorem 1. The proof in [Le] also uses the integrals

$$\int_{1}^{x} \frac{g(u)}{u^{p+1}} du \ \text{ and } \ \int_{1}^{b_i x} \frac{g(u)}{u^{p+1}} du,$$

which should be changed to

$$\int_{x_0}^{x} \frac{g(u)}{u^{p+1}} du \ \text{ and } \ \int_{x_0}^{b_i x} \frac{g(u)}{u^{p+1}} du$$

respectively. The reader can verify that the proof of the revised Theorem 1 goes through nearly unchanged from [Le].

The hypothesis of Leighton's Theorem 1 includes the condition $x_0 \geq 1/(1 - b_i)$ for $1 \leq i \leq k$, which is no longer required. It is used in [Le] only to show that the original partition is suitable for the inductive step of the proof. Our proof for the revised partition does not require that condition.

Lemma 11.1. If k is a positive integer, $a_1, \dots, a_k$ are positive real numbers, and $b_1, \dots, b_k$ are real numbers such that $0 < b_i < 1$ for each i, then there exists a unique real number p that satisfies

$$\sum_{i=1}^{k} a_i\, b_i^p = 1.$$

Proof. Define $f\colon \boldsymbol{R} \to \boldsymbol{R}$ by

$$f(x) = \sum_{i=1}^{k} a_i\, b_i^x - 1.$$

The function f has a root p because f is continuous,

$$\lim_{x \to -\infty} f(x) = +\infty > 0,$$

and

$$\lim_{x \to +\infty} f(x) = -1 < 0.$$

The function f is decreasing and hence injective because it has the negative derivative

$$f'(x) = \sum_{i=1}^{k} a_i\, b_i^x \log b_i.$$

Therefore, p is the unique root of f. □

The statement of Lemma 11.1 becomes false if we omit the requirement that $0 < b_i < 1$ for each i. For example, let $k = 2$, $a_1 = a_2 = 1$, $b_1 = 2$, and $b_2 = 1/2$. Observe that $2^p \geq 1$ if $p \geq 0$ and $(1/2)^p \geq 1$ if $p \leq 0$. Furthermore, 2^p and $(1/2)^p$ are positive for all real p. Therefore,

$$a_1 b_1^p + a_2 b_2^p = 2^p + (1/2)^p > 1$$

for every real number p. (Of course, the minimum of $2^p + (1/2)^p$ is $2^0 + (1/2)^0 = 2$.)

Lemma 11.1 justifies the following definition:

Definition. The *Akra-Bazzi exponent* of a semi-divide-and-conquer recurrence

$$(D, I, a_1, \ldots, a_k, b_1, \ldots, b_k, f, g, h_1, \ldots, h_k)$$

is the unique real number p for which

$$\sum_{i=1}^{k} a_i\, b_i^p = 1.$$

12. A Partition of the Real Numbers Into Very Dense Subsets

The main result of this section is Lemma 12.2, which will help us in our construction of an extreme counterexample to Theorem 2 of [Le]. We make use of complementary subspaces of vector spaces. If A is a vector space and B is a subspace of A, then Zorn's Lemma[1] implies the existence of a subspace C of A such that

$$A = B \oplus C.$$

Recall that A/B is the quotient space consisting of cosets $B + a$ with $a \in A$. Since A/B is isomorphic to C, we have $|A/B| = |C|$ where $|S|$ denotes the cardinality of a set S.

Suppose the scalar field is infinite, B is one-dimensional, and $A \neq B$. The subspace C is non-zero, so it contains a one-dimensional subspace isomorphic to B. Thus $|C| \geq |B|$. As $B \oplus C$ is equipotent with the Cartesian product of the infinite sets B and C, we have

$$|B \oplus C| = \max(|B|, |C|) = |C|.$$

Therefore,

$$|A| = |C| = |A/B|.$$

We follow the convention that $\boldsymbol{Q}$ represents the field of rational numbers. $\boldsymbol{R}$ is viewed as a vector space over $\boldsymbol{Q}$.

Lemma 12.1. If X is a non-empty open subset of $\boldsymbol{R}$, and V is a subspace of the rational vector space $\boldsymbol{R}$ with $|V| = |\boldsymbol{R}|$, then $|V \cap X| = |\boldsymbol{R}|$.

Proof. $V \neq 0$ implies V contains a non-zero element w, which spans a one-dimensional subspace W of V. Since W is countable, we have $W \neq V$. Then

$$|V/W| = |\boldsymbol{R}|.$$

Let v be an element of V, and define the homeomorphism $\lambda\colon \boldsymbol{R} \to \boldsymbol{R}$ by

[1] A can be infinite dimensional.

$$\lambda(t) = \frac{t - v}{w}$$

for each real number t. Then $\lambda(X)$ is a non-empty open set containing a rational number z, and

$$zw + v \in V \cap X.$$

In other words, each coset of W in V contains an element of $V \cap X$. Since the cosets are disjoint, the axiom of choice implies the existence of an injection from V/W into $V \cap X$. Therefore,

$$|\boldsymbol{R}| = |V/W| \leq |V \cap X| \leq |V| = |\boldsymbol{R}|,$$

which implies

$$|V \cap X| = |\boldsymbol{R}|.$$

□

Lemma 12.2. There exists a countably infinite partition P of $\boldsymbol{R}$ such that

$$|p \cap X| = |\boldsymbol{R}|$$

for each $p \in P$ and each non-empty open subset X of $\boldsymbol{R}$.

Proof. The vector space $\boldsymbol{R}$ over $\boldsymbol{Q}$ has $\boldsymbol{Q}$ as a one-dimensional subspace. There exists a subspace V of $\boldsymbol{R}$ such that

$$\boldsymbol{R} = \boldsymbol{Q} \oplus V.$$

Furthermore,

$$|V| = |\boldsymbol{R}|.$$

Define $P = \boldsymbol{R}/V$, which is a countably infinite partition of $\boldsymbol{R}$ into cosets of V. If $p \in P$, then $p = V + q$ for some $q \in \boldsymbol{Q}$. For each non-empty open subset X of $\boldsymbol{R}$, define the non-empty open set

$$X_q = \{x - q\colon\ x \in X\}.$$

Lemma 12.1 implies

$$|V \cap X_q| = |\boldsymbol{R}|.$$

There is a bijection from $V \cap X_q$ onto $(V + q) \cap X$ with $t \mapsto t + q$ for $t \in V \cap X_q$. Therefore,

$$|(V + q) \cap X| = |\boldsymbol{R}|,$$

i.e.,

$$|p \cap X| = |\boldsymbol{R}|.$$

□

13. Infinitely Recursive Counterexamples to Leighton's Theorem 2

In this section, we assume $x_0 \in [686,10000]$ and $f: [1, x_0] \to \boldsymbol{R}$ such that $f(x) = \Theta(1)$. Each such choice of x_0, and f determines a recurrence

$$T(x) = \begin{cases} f(x), & \text{for } 1 \le x \le x_0 \\ aT\big(bx + h(x)\big) + g(x), & \text{for } x > x_0 \end{cases}$$

where $a = b = 99/100$ and the functions $g, h: (0, \infty) \to \boldsymbol{R}$ are defined by $g(x) = 1$ and

$$h(x) = \begin{cases} 0, & \text{for } 1 \le x \le x_0 \\ \sqrt{x}, & \text{for } x > x_0. \end{cases}$$

The recurrence (actually a family of recurrences determined by the choice of x_0 and f) described above is the main subject of this section and is the object of any reference in this section to a recurrence. In Section 14, we show that the recurrence satisfies the hypothesis of Leighton's Theorem 2 with $\varepsilon = 0.74$ and $p = -1$. Meanwhile, satisfaction of the hypothesis is assumed. We will demonstrate that the recurrence is an extreme counterexample to Leighton's Theorem 2.

According to Theorem 2 of [Le],

$$T(x) = \Theta\left(x^{-1}\left(1 + \int_1^x \frac{1}{u^{-1+1}}\,du\right)\right) = \Theta\left(\frac{1}{x}\left(1 + \int_1^x du\right)\right) = \Theta(1).$$

One exact solution of the recurrence is given by $T|_{[1,x_0]} = f$ and $T(x) = 100$ for all $x > x_0$. (Corollary 13.2 will imply $bx + h(x) > x_0$ for all $x > x_0$.) This solution satisfies $T(x) = \Theta(1)$ in agreement with the formula. However, we will prove the existence of an uncountable family of other solutions that are far different from $\Theta(1)$.

Condition 3 of Leighton's Theorem 2 arguably contains a slight ambiguity. Compliance of the recurrence above with that condition is perhaps open to interpretation when $x_0 \neq 10000$. When $x_0 = 10000$, the aforementioned ambiguity is avoided and the recurrence inarguably satisfies the condition in question. We shall explain the issue after a brief digression.

For the remainder of this section, we define

$$B: (0, \infty) \to (0, \infty)$$

by

$$B(x) = bx + \sqrt{x}.$$

Of course, a function $T: [1, \infty) \to \boldsymbol{R}$ is a solution of the recurrence determined by x_0 and f if and only if $T|_{[1,x_0]} = f$ and

$$T(x) = aT\big(B(x)\big) + g(x)$$

for all $x \in (x_0, \infty)$. The function B is more convenient for some purposes than the closely related function $x \mapsto bx + h(x)$ on $[1, \infty)$, which agrees with B on (x_0, ∞). We list a few basic facts about the two functions:

Lemma 13.1.

(1) $B(10000) = 10000$.
(2) If $x \in (0{,}10000)$, then $x < B(x) < 10000$.
(3) If $y \in (10000, \infty)$, then $10000 < B(y) < y$.
(4) $0 < bz + h(z) < z$ for all $z \in [1, x_0]$.

Proof. By definition,

$$B(10000) = \frac{99}{100} \cdot 10000 + \sqrt{10000} = 9900 + 100 = 10000.$$

The function B is strictly increasing, so

$$B(x) < B(10000) = 10000.$$

Furthermore, $\sqrt{x} < \sqrt{10000} = 100$, so $x/100 < \sqrt{x}$, which implies

$$x = \frac{99x}{100} + \frac{x}{100} < \frac{99x}{100} + \sqrt{x} = B(x).$$

Similarly, $\sqrt{y} > \sqrt{10000} = 100$, so $y/100 > \sqrt{y}$, which implies

$$y = \frac{99y}{100} + \frac{y}{100} > \frac{99y}{100} + \sqrt{y} = B(y).$$

Since B is strictly increasing and $y > 10000$, we conclude that $B(y) > B(10000)$, i.e., $B(y) > 10000$.

By definition, $h(z) = 0$, so

$$bz + h(z) = bz = \frac{99}{100} z \in (0, z).$$

□

Potential ambiguity about Satisfaction of Leighton's Condition 3. Condition 3 of Leighton's Theorem 2 requires the existence of positive real numbers c_1 and c_2 such that

$$c_1 g(x) \leq g(u) \leq c_2 g(x)$$

for all $x \geq 1$ and all $u \in [bx + h(x), x]$. The function g is constant, so

$$1 \cdot g(x) \leq g(u) \leq 1 \cdot g(x)$$

for all $x, u \in (0, \infty)$.

Some authors (such as the author of this document) define $[c, d] = \{x \in \boldsymbol{R}\colon\ c \leq x \leq d\}$ for all $c, d \in \boldsymbol{R}$, so $[c, d] = \emptyset$ when $c > d$. Others do not define $[c, d]$ when $c > d$. When $x_0 = 10000$, Lemma 13.1 and agreement of B with the function $t \mapsto bt + h(t)$ on (x_0, ∞) imply $bx + h(x) < x$ for all $x \geq 1$, so Leighton's condition 3 is unambiguously satisfied. However, when $x_0 < 10000$, there exists $w \in (x_0, 10000)$. Lemma 13.1 and $w > x_0$ imply $bw + h(w) = B(w) > w$. We interpret $[bw + h(w), w]$ as the empty set, so condition 3 is vacuously satisfied when $x = w$. However, some readers may regard $[bw + h(w), w]$ as undefined and consider the mere appearance of that closed interval in condition 3 to indicate an implicit requirement that $bx + h(x) \leq x$ for all $x \geq 1$. Such readers would presumably regard our recurrence as satisfying the hypothesis of Leighton's Theorem 2 only when $x_0 = 10000$. Leighton's intention is unspecified.

Semi-divide-and-conquer recurrence is proper if and only if $\boldsymbol{x_0 = 10000}$. Observe that

$$R = (D, I, a, b, f, g, h)$$

is a semi-divide-and-conquer recurrence where $D = [1, \infty)$ and $I = (x_0, \infty)$. Lemma 13.1 implies $B(I) \subseteq D$, i.e., condition (9) of the definition of a semi-divide-and-conquer recurrence is satisfied. The remaining conditions are obvious. We also conclude from Lemma 13.1 that R is proper, i.e., $B(x) < x$ for all $x \in I$, if and only if $x_0 = 10000$.

$\boldsymbol{T(10000)}$. When $x_0 \neq 10000$, B-invariance of 10000 (Lemma 13.1) implies

$$T(10000\,) = \frac{99}{100} T(10000) + 1$$

for every solution T, i.e.,

$$T(10000) = 100.$$

The function $B\colon \boldsymbol{R}^+ \to \boldsymbol{R}^+$ has powers $B^n\colon \boldsymbol{R}^+ \to \boldsymbol{R}^+$ defined by composition of functions for all non-negative integers n. (B^0 is the identity map on $\boldsymbol{R}^+$.) Existence of negative powers of B will follow from lemma 13.3.

Corollary 13.2. The interval (x_0, ∞) is B^n–invariant for each non-negative integer n.

Proof. Let $S = (x_0, \infty)$, and suppose $x \in S$. If $x \neq 10000$, then Lemma 13.1 implies

$$B(x) > \min(x, 10000) \geq x_0,$$

so $B(x) \in S$. If instead $x = 10000 \in S$, then Lemma 13.1 implies $B(x) = x$, so again $B(x) \in S$. Therefore, $B(S) \subseteq S$. The function B^0 is the identity map on $(0, \infty)$, which contains S, so S is B^0-invariant. If n is a non-negative integer such that the interval S is B^n-invariant, then

$$B^{n+1}(S) = B\big(B^n(S)\big) \subseteq B(S) \subseteq S.$$

The proposition follows by induction. □

B–invariance of (x_0, ∞) implies the recurrence has infinite depth of recursion at all $x \in (x_0, \infty)$. We provide a C# implementation of the recurrence below. The method `T` of the `Counterexample` class throws a `StackOverflowException` for $x > x_0$. (We ignore the issue of floating-point rounding.)

```
namespace LeightonTheorem2
{
    public delegate double BaseCase(double x);

    public class Counterexample
    {
        readonly BaseCase f;
        readonly double x0;

        // valid only if 686 <= x0 <= 10000
        // and f is Big Theta of 1 on [1,x0]:

        public Counterexample(BaseCase f, double x0)
        {
            if (x0 < 686 || x0 > 10000)
            {
                throw new ArgumentOutOfRangeException();
            }
            this.f = f;
            this.x0 = x0;
        }
```

```
        public double T(double x)
        {
            if (x < 1)
            {
                throw new ArgumentOutOfRangeException();
            }
            if (x <= x0)
            {
                return f(x);
            }
            return 0.99 * T(0.99 * x + Math.Sqrt(x)) + 1;
        }
    }
}
```

Lemma 13.3. The function B is a homeomorphism from $(0, \infty)$ onto itself.

Proof. The function B is continuous. Furthermore, B is strictly increasing and therefore injective. Define the function $C: (0, \infty) \to (0, \infty)$ by

$$C(x) = \left(\frac{-1 + \sqrt{1 + 4bx}}{2b}\right)^2.$$

Then $B\big(C(x)\big) = x$ for each positive real number x, so B is surjective (hence bijective) and $B^{-1} = C$. Continuity of B^{-1} implies B is a homeomorphism. □

Alternate proof of Lemma 13.3. The general principle (which we shall not prove) is that all continuous bijections between real intervals are homeomorphisms. As before, B is injective because B is strictly increasing. The function B is continuous and satisfies

$$\lim_{x \to 0} B(x) = 0$$

and

$$\lim_{x \to \infty} B(x) = \infty.$$

The intermediate value theorem implies $B: (0, \infty) \to (0, \infty)$ is a surjection and therefore a bijection. We conclude that B is a homeomorphism. □

Among other things, Lemma 13.3 guarantees the existence of the homeomorphism B^n from $(0, \infty)$ onto itself for each integer n. As usual, powers of B refer to composition of functions, not exponentiation of function values. For example, B^2 and B^{-2} satisfy

$$B^2\big(B^{-2}(x)\big) = B^0(x) = x$$

for all $x > 0$.

Lemma 13.4. If n is an integer, then

(1) B^n is a strictly increasing function.

(2) $B^n(10000) = 10000$.

(3) The intervals $(0,10000)$ and $(10000, \infty)$ are preserved by B^n.

Proof. (1): The assertion holds for $n = 0$, because B^0 is the identify map, which is a strictly increasing function. The function B is obviously an increasing function, so B^{-1} is also increasing. Suppose k is a positive integer for which B^k and B^{-k} are increasing. The composition of increasing functions is also an increasing function, so $B^{k+1} = B \circ B^k$ and $B^{-(k+1)} = B^{-1} \circ B^k$ are increasing. The result follows by induction on $|n|$.

(2): We have $B^0(10000) = 10000$ by definition. Lemma 13.1(1) implies

$$B(10000) = 10000,$$

so

$$B^{-1}(10000) = 10000.$$

Suppose k is a non-negative integer for which $B^k(10000) = B^{-k}(10000) = 10000$. Then

$$B^{k+1}(10000) = B\left(B^k(10000)\right) = B(10000) = 10000$$

and

$$B^{-(k+1)}(10000) = B^{-1}\left(B^{-k}(10000)\right) = B^{-1}(10000) = 10000.$$

The assertion follows by induction on $|n|$.

(3): Lemma 13.3 implies $B: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ is a bijection, so $B^n: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ is also a bijection. Let $V = (0,10000)$ and $W = (10000, \infty)$. Parts (1) and (2) imply $B^n(V) \subseteq V$ and $B^n(W) \subseteq W$. By (2), we have

$$V \subset \boldsymbol{R}^+ = B^n(\boldsymbol{R}^+) = B^n(V) \cup \{B^N(10000)\} \cup B^n(W) \subseteq B^n(V) \cup \{10000\} \cup W$$

$$\subseteq B^n(V) \cup (\boldsymbol{R}^+ \backslash V),$$

which implies $V \subseteq B^n(V)$. We conclude that $B^n(V) = V$. Similarly,

$$W \subset \boldsymbol{R}^+ = B^n(\boldsymbol{R}^+) = B^n(V) \cup \{B^N(10000)\} \cup B^n(W) = V \cup \{10000\} \cup B^n(W)$$

$$\subseteq (\boldsymbol{R}^+ \backslash W) \cup B^n(W),$$

which implies $W \subseteq B^n(W)$. We conclude that $B^n(W) = W$. □

Lemma 13.5. If $m < n$ are integers, then $B^m(x) < B^n(x)$ for all $x \in (0,10000)$ and $B^m(y) > B^n(y)$ for all $y \in (10000, \infty)$.

Proof. Lemma 13.4 implies $B^j(x) \in (0,10000)$ and $B^j(y) \in (10000, \infty)$ for each integer j. Lemma 13.1 implies

$$B^m(x) < B\big(B^m(x)\big) = B^{m+1}(x)$$

and

$$B^m(y) > B\big(B^m(y)\big) = B^{m+1}(y).$$

Suppose s is positive integer such that $B^m(x) < B^{m+s}(x)$ and $B^m(y) > B^{m+s}(y)$. Lemma 13.1 implies

$$B^m(x) < B^{m+s}(x) < B\big(B^{m+s}(x)\big) = B^{m+s+1}(x)$$

and

$$B^m(y) > B^{m+s}(y) > B\big(B^{m+s}(y)\big) = B^{m+s+1}(y).$$

By induction, $B^m(x) < B^{m+t}(x)$ and $B^m(y) > B^{m+t}(y)$ for each positive integer t. In particular, $B^m(x) < B^n(x)$ and $B^m(y) > B^n(y)$ because and $n - m$ is a positive integer and $n = m + (n - m)$. □

Lemma 13.6.

(1) If $x > 0$, then
$$\lim_{n\to\infty} B^n(x) = 10000.$$

(2) If $x \in (0,10000)$, then
$$\lim_{n\to-\infty} B^n(x) = 0.$$

(3) If $x \in (10000, \infty)$, then
$$\lim_{n\to-\infty} B^n(x) = \infty.$$

Proof. (1): Lemmas 13.4 and 13.5 imply the sequence

$$x, B(x), B^2(x), \ldots$$

is monotonic and contained in the interval $J = [\min(x, 10000), \max(x, 10000)]$. The sequence converges to some $c \in J$. Then $c \in (0, \infty) = domain(B)$. Continuity of B implies the sequence

$$B(x), B^2(x), B^3(x) \ldots$$

converges to $B(c)$, i.e., $B(c) = c$. Lemma 13.1 implies 10000 is the unique fixed point of B, so $c = 10000$, i.e.,

$$\lim_{n\to\infty} B^n(x) = 10000.$$

(2) and (3): Let $x \in (0, \infty)\backslash\{10000\}$. Lemmas 13.4 and 13.5 imply

$$x, B^{-1}(x), B^{-2}(x), \ldots$$

is a monotonic sequence in $(0, \infty)\backslash\{10000\}$, so the limit

$$L = \lim_{n \to -\infty} B^n(x)$$

is defined. Furthermore, $L \in [0, \infty]$. Lemma 13.3 implies B^{-1} is continuous.

If $L \in (0, \infty)$, i.e., $L \in domain(B^{-1})$, then continuity of B^{-1} implies

$$B^{-1}(x), B^{-2}(x), B^{-3}(x), \ldots$$

converges to $B^{-1}(L)$, i.e., $L = B^{-1}(L)$. Then $B(L) = B\left(B^{-1}(L)\right) = L$, so $L = 10000$ by Lemma 13.1. However, Lemma 13.5 implies $L \neq 10000$. We conclude that $L \in \{0, \infty\}$. If $x \in (0, 10000)$, then Lemma 13.5 implies $L \neq \infty$, so $L = 0$. If $x \in (10000, \infty)$, then Lemma 13.5 implies $L \neq 0$, so $L = \infty$. □

Definition. Define the equivalence relation ~ on $(x_0, \infty)\backslash\{10000\}$ by $x \sim y$ if there exists an integer n such that $B^n(x) = y$. The equivalence class of x is denoted by $x^\sim$. For each subset X of $(x_0, \infty)\backslash\{10000\}$, let

$$X^\sim = \{x^\sim : x \in X\}.$$

A transversal of ~ is a subset of $(x_0, \infty)\backslash\{10000\}$ that contains exactly one representative of each equivalence class.

We can easily verify that ~ is an equivalence relation: $B^0(x) = x$; if $B^m(x) = y$, then $B^{-m}(y) = x$; if also $B^n(y) = z$, then $B^{m+n}(x) = z$.

We now catalog some obvious properties of equivalence classes:

Lemma 13.7. Let $x \in (x_0, \infty)\backslash\{10000\}$ and

$$S = \{n \in \boldsymbol{Z} : \ B^n(x) \in x^\sim\}.$$

Then

(1) $S = \{n \in \boldsymbol{Z} : \ B^n(x) > x_0\}$.

(2) $x^\sim$ is B-invariant.

(3) The map $\varphi: S \to x^\sim$ defined by $\varphi(n) = B^n(x)$ is a bijection.

(4) $x^{\sim}$ is countably infinite.

(5) If $x > 10000$, then $S = \boldsymbol{Z}$ and $x^{\sim} \subset (10000, \infty)$.

(6) If $x < 10000$, then $x^{\sim} \subset (x_0, 10000)$ and there exists an integer $m \leq 0$ such that

$$S = \{n \in \boldsymbol{Z} : \ n \geq m\}.$$

(7) S contains every non-negative integer.

(8) $n + 1 \in S$ for all $n \in S$.

Proof. Lemma 13.4(3) implies $B^n(x) \neq 10000$ for each integer n, so (1) follows from the definition of $\sim$.

Suppose $z \in x^{\sim}$. By definition of $\sim$, we know $z > x_0$ and $z = B^k(x)$ for some $k \in \boldsymbol{Z}$. Corollary 13.2 implies $B(z) > x_0$, i.e., $B^{k+1}(x) > x_0$. Then (1) implies $k + 1 \in S$, so $B^{k+1}(x) \in x^{\sim}$, i.e., $B(z) \in x^{\sim}$. Thus (2) is satisfied.

Lemma 13.5 implies $B^i(x) \neq B^j(x)$ when $i \neq j$ are integers, i.e., φ is injective. The map φ is surjective by definition of $\sim$. Therefore, φ is a bijection, i.e., (3) is satisfied. The set S is countable, and $x^{\sim} = \varphi(S)$, so $x^{\sim}$ is also countable.

Suppose $x > 10000$. Lemma 13.4(3) implies $B^n(x) > 10000$ for all $n \in \boldsymbol{Z}$, so $x^{\sim}$ is contained in $(10000, \infty)$. Countability of $x^{\sim}$ and uncountability of $(10000, \infty)$ imply the containment is proper. Recall that $x_0 \leq 10000$, so $B^n(x) > x_0$ for all $n \in \boldsymbol{Z}$. Statement (1) implies $S = \boldsymbol{Z}$, so (5) is satisfied.

Now suppose $x < 10000$, so $x_0 < 10000$. Lemma 13.4(3) and the definition of $\sim$ imply

$$x^{\sim} \subseteq (0,10000) \cap (x_0, \infty) = (x_0, 10000).$$

Countability of $x^{\sim}$ and uncountability of $(x_0, 10000)$ imply the containment is proper. Lemma 13.5 implies

$$\ldots < B^{-2}(x) < B^{-1}(x) < x < B(x) < B^2(x) < \cdots.$$

Lemma 13.6(2) implies

$$\lim_{n \to -\infty} B^n(x) = 0 < x_0 < x = B^0(x),$$

so (6) follows from (1).

Statements (5) and (6) imply S is countably infinite, so (3) implies (4). Statements (5) and (6) imply (7) and (8) □

Lemma 13.8. If $x \in (x_0, \infty)\backslash\{10000\}$ and $y \in \boldsymbol{R}$, then there exists exactly one function $\lambda: x^{\sim} \to \boldsymbol{R}$ such that $\lambda(x) = y$ and

$$\lambda(z) = a\lambda\big(B(z)\big) + 1$$

for all $z \in x^{\sim}$.

Proof. (Lemma 13.7(2) implies $B(z) \in x^{\sim} = domain(\lambda)$ for all $z \in x^{\sim}$.) Recall that $\boldsymbol{N}$ denotes the set of non-negative integers. Inductively define $u: \boldsymbol{N} \to \boldsymbol{R}$ by $u(0) = y$ and

$$u(n+1) = \frac{u(n) - 1}{a},$$

so

$$u(n) = au(n+1) + 1$$

for all $n \in \boldsymbol{N}$. Recursively define $w: \boldsymbol{Z}^+ \to \boldsymbol{R}$ by

$$w(1) = au(0) + 1$$

and

$$w(n) = aw(n-1) + 1$$

for all $n > 1$. Let $\boldsymbol{Z}^-$ denote the set of negative integers and define $v: \boldsymbol{Z}^- \to \boldsymbol{R}$ by $v(n) = w(-n)$, so

$$v(-1) = au(0) + 1$$

and

$$v(n) = aw(-n-1) + 1 = av(n+1) + 1$$

for all $n < -1$. Define $r: \boldsymbol{Z} \to \boldsymbol{Z}$ by $r|_{\boldsymbol{N}} = u$ and $r|_{\boldsymbol{Z}^-} = v$. Then $r(0) = u(0) = y$ and

$$r(n) = ar(n+1) + 1$$

for all $n \in \boldsymbol{Z}$. Let

$$S = \{n \in \boldsymbol{Z} :\ B^n(x) \in x^{\sim}\},$$

and define $\varphi: S \to x^{\sim}$ by $\varphi(n) = B^n(x)$. The function φ is a bijection by Lemma 13.7(3) and has an inverse $\varphi^{-1}: x^{\sim} \to S$. Define $\lambda: x^{\sim} \to \boldsymbol{R}$ by $\lambda(z) = r\big(\varphi^{-1}(z)\big)$, so

$$\lambda(x) = r(0) = y.$$

If $z \in x^{\sim}$ and $n = \varphi^{-1}(z)$, then $n \in S$. Lemma 13.7(8) implies $n + 1 \in S$, so

$$\varphi(n+1) = B^{n+1}(x) = B\big(B^n(x)\big) = B(z)$$

and

$$\lambda(z) = r(n) = ar(n+1) + 1 = a\lambda\big(B(z)\big) + 1.$$

Now suppose $\mu: x^{\sim} \to \boldsymbol{R}$ such that $\mu(x) = y$ and

$$\mu(z) = a\mu\big(B(z)\big) + 1$$

for all $z \in x^{\sim}$. Let

$$W = \{n \in \boldsymbol{Z} : \text{either } n \notin S \text{ or } \lambda\big(B^n(x)\big) = \mu\big(B^n(x)\big)\}$$

and

$$W^* = \{n \in \boldsymbol{N} \cap W : -n \in W\}.$$

We know $0 \in W$ because $B^0(x) = x$ and

$$\lambda(x) = y = \mu(x).$$

More specifically, $0 \in \boldsymbol{N} \cap W$. Of course, $-0 = 0 \in W$, so $0 \in W^*$. Let $n \in W^*$, so $n \in \boldsymbol{N} \cap W$. Lemma 13.7(7) implies $\boldsymbol{N} \subseteq S$, so $n \in S \cap W$, which implies

$$\lambda\big(B^n(x)\big) = \mu\big(B^n(x)\big),$$

i.e.,

$$a\lambda\big(B^{n+1}(x)\big) + 1 = a\mu\big(B^{n+1}(x)\big) + 1.$$

Then

$$\lambda\big(B^{n+1}(x)\big) = \mu\big(B^{n+1}(x)\big),$$

which implies $n + 1 \in W$. Of course, $n + 1 \in \boldsymbol{N}$ because $n \in \boldsymbol{N}$, so $n + 1 \in \boldsymbol{N} \cap W$. Suppose $-(n + 1) \in S$, so $-n \in S$ by Lemma 13.7(8). We know $-n \in W$ because $n \in W^*$. Now

$$\lambda\big(B^{-n-1}(x)\big) = a\lambda\big(B^{-n}(x)\big) + 1 = a\mu\big(B^{-n}(x)\big) + 1 = \mu\big(B^{-n-1}(x)\big),$$

so $-(n + 1) \in W$, which implies $n + 1 \in W^*$. Now suppose instead that $-(n + 1) \notin S$. Then $-(n + 1) \in W$ by definition and again $n + 1 \in W^*$. By induction, $\boldsymbol{N} \subseteq W^*$, i.e., $W^* = \boldsymbol{N}$. Then $W = \boldsymbol{Z}$, so

$$\lambda\big(B^n(x)\big) = \mu\big(B^n(x)\big)$$

for all $n \in S$. Surjectivity of φ implies $\lambda(z) = \mu(z)$ for all $z \in x^{\sim}$. Thus $\lambda = \mu$. □

Lemma 13.9. If S is a transversal of the equivalence relation $\sim$, then each real-valued function on S has a unique extension to a solution of the recurrence.

Proof. Let $t: S \to \boldsymbol{R}$. Lemma 13.8 implies that for each $x \in S$, there exists exactly one function $\lambda_x: x^{\sim} \to \boldsymbol{R}$ with $\lambda_x(x) = t(x)$ such that

$$\lambda_x(z) = a\lambda_x\big(B(z)\big) + 1$$

for all $z \in x^{\sim}$. (Recall that $x^{\sim}$ is B-invariant by Lemma 13.7(2), i.e., $B(z) \in domain(\lambda_x)$ for all $z \in x^{\sim}$.) Let $W = (x_0, \infty)\backslash\{10000\}$. The set W is a disjoint union of equivalence classes of elements of S:

$$W = \dot{\bigcup_{x \in S}} x^{\sim}.$$

There exists a unique real-valued function $\varphi: W \to \boldsymbol{R}$ with $\varphi|_{x^{\sim}} = \lambda_x$ for all $x \in S$. The function φ satisfies $\varphi(x) = \lambda_x(x) = t(x)$ for all such x, i.e., $\varphi|_S = t$. For all $z \in W$, there exists $x \in S$ such that $z \in x^{\sim}$. Then $B(z) \in x^{\sim}$, so $z, B(z) \in W = domain(\varphi)$. Furthermore,

$$\varphi(z) = \lambda_x(z) = a\lambda_x\big(B(z)\big) + 1 = a\varphi\big(B(z)\big) + 1.$$

Define the function $T: [1, \infty) \to \boldsymbol{R}$ by $T|_{[1,x_0]} = f$, $T|_W = \varphi$, and if $x_0 \neq 10000$, i.e., $x_0 < 10000$, define $T(10000) = 100$. For all $z \in W$, we have

$$T(z) = \varphi(z) = a\varphi\big(B(z)\big) + 1 = aT\big(B(z)\big) + 1.$$

If $x_0 \neq 10000$, then

$$T(10000) = 100 = \tfrac{99}{100} \cdot 100 + 1 = aT(10000) + 1.$$

Therefore, T is a solution of the recurrence (regardless of whether $x_0 = 10000$). Furthermore,

$$T|_S = (T|_W)|_S = \varphi|_S = t.$$

Let T^* be any solution of the recurrence that satisfies $T^*(S) = t$. Then

$$T^*(x) = t(x)$$

for all $x \in S$, and

$$T^*(z) = aT^*\big(B(z)\big) + 1$$

for all $z \in x^{\sim}$, so $T^*|_{x^{\sim}} = \lambda_x$. Therefore, $T^*|_W = \varphi = T|_W$. By definition of the recurrence,

$$T^*|_{[1,x_0]} = f = T|_{[1,x_0]}.$$

If $x_0 \neq 10000$, then $T^*(10000) = 100 = T(10000)$. The functions T and T^* have domain

$$[1, \infty) = W \cup [1, x_0] \cup \{10000\}.$$

Therefore, $T^* = T$. (Of course, $[1, \infty)$ is the simpler union $W \cup [1, x_0]$ if $x_0 = 10000$.)

□

Each equivalence class represents a degree of freedom in the recurrence. Observe that in Lemma 13.9 neither the real-valued function on a transversal nor its extension to a solution of the recurrence is required to be positive or non-negative.

Definition. A subset S of $(x_0, \infty)\backslash\{10000\}$ is *dependent* (relative to $\sim$) if there exist distinct $x, y \in S$ such that $x \sim y$. Otherwise, S is *independent*.

Of course, the transversals of $\sim$ are precisely the maximum independent subsets of the punctured interval $(x_0, \infty)\backslash\{10000\}$. By Zorn's Lemma, every independent subset of $(x_0, \infty)\backslash\{10000\}$ can be extended to a transversal of $\sim$.

Corollary 13.10. If S is an independent subset of $(x_0, \infty)\backslash\{10000\}$ relative to $\sim$, then each real-valued function on S can be extended to a solution of the recurrence.

Proof. Let $t: S \to \boldsymbol{R}$. The independent set S is contained in a transversal S^* of $\sim$. The function t can be extended to a function $t^*: S \to \boldsymbol{R}$, which can be extended to a solution T of the recurrence by Lemma 13.9. Of course, T is an extension of t. □

Lemma 13.11. There exists a transversal S of $\sim$ with $\sup S = \infty$.

Proof. Lemma 13.4(3) implies the interval $J = (10000, \infty)$ is the union of equivalence classes. Lemma 3.7(4) implies each equivalence class is countable. Uncountability of J implies J contains infinitely many equivalence classes. There exists a countable infinite set

$$U = \{C_1, C_2, C_3, \dots\}$$

of disjoint equivalence classes that are contained in J. Define $C_n^* = C_n \cap (n, \infty)$ for all $n \in \boldsymbol{Z}^+$, and let

$$U^* = \{C_1^*, C_2^*, C_3^*, \dots\}.$$

Lemma 13.6(3) implies $C_n^* \neq \emptyset$ for all $n \in \boldsymbol{Z}^+$. The axiom of choice implies the existence of a function $r: U^* \to J$ such that $r(C_n^*) \in C_n^*$ for all n, so $r(C_n^*) > n$. The set $r(U^*)$ is independent relative to $\sim$. Zorn's Lemma implies $r(U^*)$ is contained in a transversal S of $\sim$. Furthermore, $\sup r(U^*) = \infty$, so $\sup S = \infty$. □

The proof above of Lemma 13.11 uses Zorn's lemma and the equivalent axiom of choice. A constructive proof is provided after Lemma 13.14.

Corollary 13.12. The recurrence has a solution T that agrees with the exponential function on an unbounded set. In particular, T is not $\Theta(1)$.

Proof. By Lemma 13.11, there exists a transversal S of $\sim$ with $\sup S = \infty$. Lemma 13.9 implies there exists a solution T of the recurrence with

$$T(x) = e^x$$

for all $x \in S$. □

The preceding proposition shows that the recurrence does not satisfy the conclusion of Theorem 2 of [Le]. We shall establish the existence of a particularly wild solution of the recurrence.

Lemma 13.13. If $x \neq 10000$ is a positive real number and

$$0 < y < 10000 < z < \infty,$$

then there exists a unique integer n that satisfies

$$B^n(x) \in \big[y, B(y)\big) \cup (B(z), z].$$

Proof. Lemma 13.1 implies

$$y < B(y) < 10000 < B(z) < z.$$

so $\big[y, B(y)\big) \subset (0{,}10000)$ and $(B(z), z] \subset (10000, \infty)$.

Suppose $x < 10000$. Lemma 13.6 implies the set $\{i \in \boldsymbol{Z}\colon\ B^i(x) \geq y\}$ of integers is non-empty and bounded below and hence has a least element n. Then

$$B^{n-1}(x) < y \leq B^n(x).$$

B is an increasing function by Lemma 13.4(1), so

$$B^n(x) = B\big(B^{n-1}(x)\big) < B(y) \leq B\big(B^n(x)\big) = B^{n+1}(x).$$

Observe that

$$B^n(x) \in \big[y, B(y)\big) \subset \big[y, B(y)\big) \cup (B(z), z].$$

Let m be any integer other than n. Lemma 13.5 implies

$$B^m(x) \leq B^{n-1}(x) < y$$

when $m < n$, and

$$B(y) \leq B^{n+1}(x) \leq B^m(x)$$

when $m > n$. Therefore, $B^m(x) \notin \big[y, B(y)\big)$. Lemma 13.4(3) implies $B^m(x) < 10000$, which implies $B^m(x) \notin (B(z), z]$, so

$$B^m(x) \notin \big[y, B(y)\big) \cup (B(z), z].$$

Now suppose instead that $x > 10000$. Lemma 13.6 implies the set of integers $\{i \in \boldsymbol{Z}\colon B^i(x) \leq z\}$ is non-empty and bounded below and hence has a least element n. Then

$$B^n(x) \leq z < B^{n-1}(x).$$

Since B is an increasing function, we conclude that

$$B^{n+1}(x) = B\big(B^n(x)\big) \leq B(z) < B\big(B^{n-1}(x)\big) = B^n(x).$$

Observe that

$$B^n(x) \in (B(z), z] \subset \big[y, B(y)\big) \cup (B(z), z].$$

Again let m be any integer other than n. Lemma 13.5 implies

$$z < B^{n-1}(x) \leq B^m(x)$$

when $m < n$, and

$$B^m(x) \leq B^{n+1}(x) \leq B(z)$$

when $m > n$. Therefore, $B^m(x) \notin (B(z), z]$. Lemma 13.4 implies $B^m(x) > 10000$, which implies $B^m(x) \notin \big[y, B(y)\big)$, so

$$B^m(x) \notin \big[y, B(y)\big) \cup (B(z), z].$$

□

Lemma 13.14. If $x_0 = 10000$, then $(B(z), z]$ is a transversal of $\sim$ for all $z > 10000$. If $x_0 \neq 10000$, then $\big[y, B(y)\big) \cup (B(z), z]$ is a transversal of $\sim$ for all $y \in (x_0, 10000)$ and all $z > 10000$.

Proof. Let $z > 10000$ and

$$J = (x_0, \infty)\backslash\{10000\},$$

so

$$(B(z), z] \subset (10000, \infty) \subseteq J$$

by Lemma 13.1. For all $t \in (0{,}10000)$, define

$$S(t) = \big[t, B(t)\big) \cup (B(z), z].$$

Lemma 13.13 implies that for each $x \in J$ and each $t \in (0{,}10000)$ there exists exactly one integer n for which $B^n(x) \in S(t)$. If $B^n(x) \in J$, then $B^n(x) \in x^\sim$ and $|x^\sim \cap S(t)| = 1$. If $B^n(x) \notin J$, then $B^n(x) \notin x^\sim$ and $x^\sim \cap S(t) = \emptyset$.

Suppose $x_0 \neq 10000$, so $x_0 < 10000$. Let $y \in (x_0, 10000)$, so $S(y) \subset J$ by Lemma 13.1. Then $|x^\sim \cap S(y)| = 1$ for all $x \in J$, i.e., $S(y)$ is a transversal of the equivalence relation $\sim$.

Now suppose $x_0 = 10000$, so $J = (10000, \infty)$. The interval J is B^m-invariant for all $m \in \boldsymbol{Z}$ by Lemma 13.4(3). Let $u \in (0{,}10000)$. If $x \in J$ and $n \in \boldsymbol{Z}$ with $B^n(x) \in S(u)$, then $x^n \in J \cap S(u)$, so $x^n \in x^\sim$ and $|x^\sim \cap J \cap S(u)| = 1$. The set $J \cap S(u)$ is contained in J, so $J \cap S(u)$ is a transversal. Lemma 13.1 implies

$$J \cap S(u) = (B(z), z],$$

so $(B(z), z]$ is a transversal of $\sim$. □

Constructive proof of Lemma 13.11. Let $z > 10000$. Lemma 13.1(3) implies $10000 < B(z) < z$. If $x_0 = 10000$, define $S = (B(z), z]$; otherwise let $t \in (x_0, 10000)$ and define

$$S = [t, B(t)) \cup (B(z), z].$$

Lemma 13.14 implies S is a transversal of the equivalence relation $\sim$. For each positive integer n, define

$$x_n = B(z) + \frac{z - B(z)}{n}$$

and

$$y_n = B^{-n}(x_n).$$

Let

$$X = \{x_n : n \in \boldsymbol{Z}^+\},$$

$$Y = \{y_n : n \in \boldsymbol{Z}^+\},$$

and

$$S^* = (S - X) \cup Y.$$

Observe that

$$x_n \in (B(z), z] = S \cap (10000, \infty).$$

Lemma 13.4(3) implies $y_n \in (10000, \infty)$, so $y_n \in (x_0, \infty)$, which imples $y_n \in x_n^\sim$, i.e., $x_n^\sim = y_n^\sim$. Therefore, S^* is a transversal of $\sim$. Lemma 13.4(1) combines with $x_n > B(z)$ to imply $y_n > B^{1-n}(z)$ for all $n \in \boldsymbol{Z}^+$. Lemma 13.6(3) implies

$$\lim_{n\to\infty} B^{1-n}(z) = \infty,$$

so

$$\lim_{n\to\infty} y_n = \infty,$$

which combines with $Y \subset S^*$ to imply $\sup S^* = \infty$. □

Lemma 13.15. Let U be the set of all non-empty open subintervals of (x_0, ∞) with rational endpoints. There exists an independent (relative to $\sim$) set Γ along with a partition Π of Γ and a bijection $\pi: U \to \Pi$ such that $\pi(u) \subseteq u$ and $|\pi(u)| = |\boldsymbol{R}|$ for all $u \in U$.

Proof. Define $u^* = u \backslash \{10000\}$ for all $u \in U$, so u^* is non-empty for all such u. Let

$$U^* = \{u^* : u \in U\}.$$

The axiom of choice implies the existence of a function λ on U^* such that $\lambda(u^*) \in u^*$ for all $u^* \in U^*$. Define the function $c: U \to (x_0, \infty)$ by $c(u) = \lambda(u^*)$. Then $c(u) \in u$ and $c(u) \neq 10000$ for all $u \in U$.

Lemma 13.14 implies the existence of a transversal S of $\sim$ that satisfies is $I \subset S \subset \bar{I}$ where I is the interior of S and $\bar{I}$ is the closure of I. There exists a function $n: U \to \boldsymbol{Z}$ satisfying

$$B^{n(u)}\big(c(u)\big) \in S \subset \bar{I}$$

for all $u \in U$. By Lemma 13.3, the function $B^{n(u)}: (0, \infty) \to (0, \infty)$ is a homeomorphism, so $B^{n(u)}(u)$ is an open set containing the element $B^{n(u)}\big(c(u)\big)$ of $\bar{I}$. Therefore, the open set $I \cap B^{n(u)}(u)$ is non-empty.

By Lemma 12.2, there exists a countably infinite partition P of $\boldsymbol{R}$ such that $|r \cap q| = |\boldsymbol{R}|$ for each element r of P, and each non-empty open subset q of $\boldsymbol{R}$. Since U is also countably infinite, there exists a bijection $\alpha: U \to P$. Let $\delta: U \to 2^I$ be the function defined by

$$\delta(u) = \alpha(u) \cap I \cap B^{n(u)}(u)$$

for each $u \in U$, so

$$\delta(u) \subseteq I \subset S \subset (x_0, \infty)\backslash\{10000\}$$

and $|\delta(u)| = |\boldsymbol{R}|$.

The elements of P are disjoint, so $\alpha(u)$ and $\alpha(v)$ are disjoint for every pair of distinct elements u and v of U. Therefore, $\delta(u)$ and $\delta(v)$ are also disjoint for such u and v. Since $\delta(u) \subset S$ and $\delta(v) \subset S$, we conclude that

$$\big(\delta(u)\big)^{\sim} \cap \big(\delta(v)\big)^{\sim} = \emptyset.$$

Define $\pi: U \to 2^{(0,\infty)}$ by

$$\pi(u) = B^{-n(u)}\big(\delta(u)\big)$$

for all $u \in U$, so

$$\pi(u) \subseteq B^{-n(u)}\left(B^{n(u)}(u)\right) = u \subset (x_0, \infty).$$

Lemma 13.4 and $10000 \notin \delta(u)$ imply $10000 \notin \pi(u)$, i.e.,

$$\pi(u) \subseteq u^* \subset (x_0, \infty)\backslash\{10000\}.$$

The function $B^{-n(u)}: (0, \infty) \to (0, \infty)$ is a bijection by Lemma 13.3, so

$$|\pi(u)| = |\delta(u)| = |\boldsymbol{R}|.$$

In particular, $\pi(u) \neq \emptyset$. For each pair of distinct elements u and v of U, we have

$$\pi(u) \cap \pi(v) \subseteq \big(\pi(u)\big)^{\sim} \cap \big(\pi(v)\big)^{\sim} = \big(\delta(u)\big)^{\sim} \cap \big(\delta(v)\big)^{\sim} = \emptyset,$$

so $\pi(u) \neq \pi(v)$. Thus π is injective. Define $\Pi = \pi(U)$, so $\pi: U \to \Pi$ is a bijection from U onto Π. Also define

$$\Gamma = \bigcup_{u \in \mathrm{U}} \pi(u),$$

so $\Gamma \subseteq (x_0, \infty)\backslash\{10000\}$ and Π is a partition of Γ. Let x and y be distinct elements of Γ. There exist $u, w \in U$ such that $x \in \pi(u)$ and $y \in \pi(w)$, so

$$B^{n(u)}(x) \in \delta(u) \subset S$$

and

$$B^{n(w)}(y) \in \delta(w) \subset S.$$

If $u = w$, then $B^{n(u)}(x)$ and $B^{n(u)}(y)$ are distinct elements of S because $B^{n(u)}$ is injective, so

$$x^{\sim} = \left(B^{n(u)}(x)\right)^{\sim} \neq \left(B^{n(u)}(y)\right)^{\sim} = y^{\sim}.$$

If $u \neq w$, then

$$x^{\sim} \cap y^{\sim} \subseteq \left(\pi(u)\right)^{\sim} \cap \left(\pi(w)\right)^{\sim} = \emptyset,$$

so $x^{\sim} \neq y^{\sim}$. Therefore, the set Γ is an independent subset of $(x_0, \infty)\backslash\{10000\}$ relative to the equivalence relation $\sim$. □

We are now ready to show the existence of an extreme counterexample to Theorem 2 of [Le] in the form of an erratic solution to the recurrence at the beginning of this section.

Lemma 13.16. The recurrence has a solution T such that

$$T(X) = \boldsymbol{R}$$

for each non-empty open set X in (x_0, ∞). In particular, the graph of T is dense in the open half plane defined by the inequality $x > x_0$.

Proof. Let U, Γ, Π, and π be as in lemma 13.15. Then $|C| = |\boldsymbol{R}|$ for all $C \in \Pi$. For all such C, let C^* be the set of bijections from C onto $\boldsymbol{R}$, so $C^* \neq \emptyset$. Define

$$S = \{C^* : C \in \Pi\}.$$

By the axiom of choice, there exists a function β on S such that $\beta(r) \in r$ for all $r \in$ S. Define a function α on Π by $\alpha(C) = \beta(C^*)$ for all $C \in \Pi$. Then $\alpha(C) \in C^*$, i.e., $\alpha(C)$ is a bijection from C onto $\boldsymbol{R}$ for all such C. Since Π is a partition of Γ, there exists a function $t: \Gamma \to \mathbf{R}$ such that

$$t|_C = \alpha(C)$$

for all $C \in \Pi$. Corollary 13.10 implies t can be extended to a solution T of the recurrence. Suppose X is a non-empty open set in (x_0, ∞), so there exists $w \in U$ such that $w \subseteq X$. Let $D = \pi(w)$, so $D \in \Pi$ and $D \subseteq w$. Then

$$\boldsymbol{R} = \alpha(D)(D) = t(D) = T(D) \subseteq T(w) \subseteq T(X) \subseteq \boldsymbol{R}.$$

Therefore, $T(X) = \boldsymbol{R}$. □

Uncountably many choices for T in Lemma 3.16. Examination of the proof of Lemma 13.16 reveals that there are uncountably many choices for the solution T with the specified properties: Let $C \in \Pi$. The set C^* is uncountable, so there are uncountably many choice functions φ on S that satisfy $\varphi(C^*) \in C^*$ and $\varphi(r) = \beta(r)$ for all $r \in S \backslash \{C^*\}$. Each such choice function determines a solution T_φ with the specified properties. Two choice functions determine the same solution if and only if the choice functions are equal. Therefore, there are uncountably many solutions that satisfy the conclusion of Lemma 3.16.

Theorem 2 of [Le] asserts that solutions to the recurrence at the beginning of this section must be $\Theta(1)$. However, the solution T described in Lemma 3.16 is extremely different from $\Theta(1)$ and is not even asymptically non-negative. The solution is wildly unconstrained everywhere outside the domain of the base case.

We claim that for every continuous function $k: (x_0, \infty) \to \boldsymbol{R}$ and every non-empty open subset O of (x_0, ∞), there exists $z_1, z_2 \in O$ such that

$$T(z_1) > |k(z_1)|$$

and

$$T(z_2) < -|k(z_2)|.$$

To verify the claim, observe that O contains a non-empty compact interval $[L, M]$. Continuity of k implies k is bounded on $[L, M]$, i.e., there exists $Y \in \boldsymbol{R}$ such that $|k(x)| \leq Y$ for all $x \in [L, M]$. By choice of T, there exist $z_1, z_2 \in (L, M) \subseteq [L, M]$ such that

$$T(z_1) > Y \geq |k(z_1)|$$

and

$$T(z_2) < -Y \leq -|k(z_2)|.$$

For example, there exists $u, w \in O$ such that

$$T(u) > SuperExp(u)$$

and

$$T(w) < -SuperExp(w)$$

where $SuperExp: \boldsymbol{R} \to \boldsymbol{R}^+$ is defined by

$$SuperExp(x) = e^{e^{e^{e^{e^{e^{e^{e^{e^{e^{x}}}}}}}}}}.$$

We have demonstrated that Theorem 2 of [Le] is false, but is all lost? In later sections, we describe replacements for the theorem and provide proofs that are adapted from the arguments of [Le]. The new propositions have arguably simpler hypotheses. Modest restrictions on the allowed recurrences imply that a strong form of the Akra-Bazzi formula is satisfied. Recurrences admissible by the new theorems do not model recursive algorithms that begin execution next week but terminated before the big bang ($T(z)$ large negative).

Our main goals for this family of counterexamples to Leighton's Theorem 2 have been accomplished. However, we continue our analysis. We are especially interested in solutions that are $\Theta(1)$.

Lemma 13.17. If T is a solution of the recurrence, then the formula

$$T(x) = a^n\big(T\big(B^n(x)\big) - 100\big) + 100$$

is satisfied for $x > x_0$ and $n \in \boldsymbol{Z}$ when x and n satisfy any of the following conditions:

(1) $n \geq 0$.
(2) $B^n(x) > x_0$.
(3) $x \geq 10000$.

Proof. Recall that $a = 99/100$. Observe that $a^0 = 1$ and B^0 is the identity function on the interval $(0, \infty)$, so

$$T(x) = a^0\big(T\big(B^0(x)\big) - 100\big) + 100.$$

for all $x > x_0$ (indeed, the identity is true for all x in the domain of T, i.e., $x \geq 1$). Suppose $n \geq 0$ is an integer such that

$$T(x) = a^n\big(T\big(B^n(x)\big) - 100\big) + 100$$

for all $x > x_0$. Corollary 13.2 implies $B^n(x) > x_0$ for all such x, so

$$T\big(B^n(x)\big) = aT\big(B^{n+1}(x)\big) + 1,$$

which implies

$$T(x) = a^n\big(aT\big(B^{n+1}(x)\big) - 99\big) + 100$$

$$= a^{n+1}\big(T\big(B^{n+1}(x)\big) - 100\big) + 100.$$

By induction, the formula is satisfied when condition (1) is true. Now suppose instead that (2) is true, i.e., $B^n(x) > x_0$. We may assume (1) is false, i.e., $n < 0$. Then $-n > 0$, and

$$T\big(B^n(x)\big) = a^{-n}\left(T\left(B^{-n}\big(B^n(x)\big)\right) - 100\right) + 100$$

$$= a^{-n}(T(x) - 100) + 100.$$

Then

$$T(x) = a^n\big(T\big(B^n(x)\big) - 100\big) + 100$$

as required. Suppose (3) is true. If $x > 10000$, then Lemma 13.4(3) implies

$$B^n(x) > 10000 \geq x_0;$$

if $x = 10000$, then Lemma 13.4(2) implies

$$B^n(x) = x > x_0.$$

Condition (2) is true, so the formula is satisfied. □

Corollary 13.18. Suppose T is a solution of the recurrence and either $x > 10000$ or $x = 10000 \neq x_0$. Then

$$\lim_{n \to -\infty} T(B^n(x)) = 100.$$

Proof. We conclude from $x_0 \in [686,10000]$ that $x > x_0$, which combines Lemma 13.17 and $x \geq 10000$ to imply

$$T(x) = a^n\big(T(B^n(x)) - 100\big) + 100,$$

so

$$T(B^n(x)) = a^{-n}(T(x) - 100) + 100.$$

Recall that $a = 99/100$, so

$$\lim_{n \to -\infty} a^{-n} = \lim_{n \to \infty} a^n = 0,$$

which implies

$$\lim_{n \to -\infty} T(B^n(x)) = 100.$$

□

Restriction to an equivalence class. If T is a solution of the recurrence and $x > 10000$, then Corollary 13.18 implies $T(z)$ approaches 100 as z in $x^{\sim}$ approaches ∞. (See Lemmas 13.5 and 13.6.)

Dangerous Bend. Although every solution T has a $\Theta(1)$ restriction to each equivalence class in $(10000, \infty)$, Corollary 13.12 (also Lemma 13.16) implies the existence of solutions that are not $\Theta(1)$. Each such solution has non-uniform convergence on equivalence classes in $(10000, \infty)$.

Lemma 13.19. Let T be a solution of the recurrence. Either $T(x) = 100$ for all $x > 10000$ or

$$\limsup_{x \to 10000^+} |T(x)| = \infty.$$

If $x_0 < 10000$, then either $T(x) = 100$ for all $x \in (x_0, 10000)$ or

$$\limsup_{x \to 10000^-} |T(x)| = \infty.$$

Proof. Suppose $w > x_0$ such that $T(w) \neq 100$. If n is any non-negative integer, then Lemma 13.17 implies

$$T(w) = a^n\big(T\big(B^n(w)\big) - 100\big) + 100,$$

i.e.,

$$T\big(B^n(w)\big) = a^{-n}(T(w) - 100) + 100.$$

The limit

$$\lim_{n\to\infty} a^{-n} = \infty$$

implies

$$\lim_{n\to\infty}\big|T\big(B^n(w)\big)\big| = \infty.$$

Lemma 13.6(1) implies

$$\lim_{n\to\infty} B^n(w) = 10000.$$

If $w > 10000$, then Lemma 13.4(3) implies $B^n(w) > 10000$, so

$$\lim_{x\to 10000^+} \sup |T(x)| = \infty.$$

If $w < 10000$, then Lemma 13.4(3) implies $B^n(w) < 10000$, so

$$\lim_{x\to 10000^-} \sup |T(x)| = \infty.$$

□

Lemma 13.20. Let T be a solution of the recurrence. The following statements are either all true or all false:

(1) $T(x) = \Theta(1)$.
(2) $T(x)$ approaches 100 as x approaches ∞.
(3) T is bounded on $(B(x), x)$ for *some* $x > 10000$.
(4) T is bounded on $(B(x), x)$ for *all* $x > 10000$.
(5) T is bounded on all bounded subsets of (x, ∞) for *some* $x > 10000$.
(6) T is bounded on all bounded subsets of (x, ∞) for *all* $x > 10000$.

(Of course, (5) could specify $x \geq 1$ instead of $x > 10000$, which is specified for symmetry with (6).)

Proof. We will show (6) ⇒ (5) ⇒ (4) ⇒ (3) ⇒ (2) ⇒ (1) ⇒ (3) ⇒ (6) and the proposition will be proved. If (2) is satisfied, then

$$50 < T(x) < 150$$

for all sufficiently large x, so $T(x) = \Theta(1)$, i.e., (2) implies (1). The interval $(10000, \infty)$ is non-empty, so (4) implies (3) and (6) implies (5).

Suppose $T(x) = \Theta(1)$, which implies T is bounded on (c, ∞) for some real $c \geq 1$. By Lemma 13.3, there exists $u > 0$ such that $B(u) > \max(c, 10000)$. Lemma 13.4 implies

$u > 10000$. The interval $(B(u), u)$ is contained in (c, ∞), so T is bounded on $(B(u), u)$, i.e., (1) implies (3).

Now suppose (5) is satisfied, i.e., T is bounded on all bounded subsets of (z, ∞) for some $z > 10000$. Let $y > 10000$ and define $Y = (B(y), y)$. Lemma 13.1(3) implies $y > B(y) > 10000$, so Y is non-empty and contained in $(10000, \infty)$. Lemma 13.6(3) implies $B^{m+1}(y) > z$ for some integer m. Lemmas 13.3 and 13.4(1) imply

$$B^m(Y) = \left(B^{m+1}(y), B^m(y)\right),$$

so $B^m(Y)$ is a bounded subset of (z, ∞), which implies T is bounded on $B^m(Y)$. Lemma 13.17 implies

$$\inf T(Y) = a^m \cdot \left(\inf T\left(B^m(Y)\right) - 100\right) + 100 > -\infty$$

and

$$\sup T(Y) = a^m \cdot \left(\sup T\left(B^m(Y)\right) - 100\right) + 100 < \infty,$$

i.e., T is bounded on Y, so (5) implies (4).

We now assume (3) is satisfied, i.e., T is bounded on $(B(w), w)$ for some $w > 10000$. Lemma 13.1(3) implies $B(w) < w$. Then

$$\sup_{x \in (B(w), w]} |T(x)| = \max\left(|T(w)|, \sup_{x \in (B(w), w)} |T(x)|\right) < \infty.$$

In particular, T is bounded on $(B(w), w]$. Lemmas 13.4(3) and 13.5 imply

$$10000 < \cdots < B^3(w) < B^2(w) < B(w) < w < B^{-1}(w) < B^{-2}(w) < B^{-3}(w) < \cdots.$$

Lemma 13.6 says

$$\lim_{n \to \infty} B^n(w) = 10000$$

and

$$\lim_{n \to -\infty} B^n(w) = \infty.$$

For each integer n, define

$$S_n = (B^{n+1}(w), B^n(w)],$$

so $S_n \subset (10000, \infty)$. Let

$$L_n = \inf T(S_n)$$

and

$$U_n = \sup T(S_n),$$

so $L_0 > -\infty$, and $U_0 < \infty$. Lemmas 13.3 and 13.4(1) imply $B^{-n}(S_n) = S_0$. Lemma 13.17 implies

$$L_n = a^{-n} \cdot (L_0 - 100) + 100 > -\infty$$

and

$$U_n = a^{-n} \cdot (U_0 - 100) + 100 < \infty.$$

Observe that

$$\lim_{n\to-\infty} a^{-n} = \lim_{n\to\infty} a^{n} = 0,$$

so

$$\lim_{n\to-\infty} L_n = \lim_{n\to-\infty} U_n = 100.$$

Let $\varepsilon > 0$. There exists $r \in \boldsymbol{Z}$ such that $L_n, U_n \in (100-\varepsilon, 100+\varepsilon)$ for all $n < r$. Then $|T(x) - 100| < \varepsilon$ for all

$$x \in \bigcup_{n<r} S_n = (B^r(w), \infty).$$

Therefore,

$$\lim_{x\to\infty} T(x) = 100,$$

i.e., (3) implies (2). Now let $v > 10000$, and let Q be any bounded subset of (v, ∞), so $\inf Q > 10000$ There exists integers α and β with $\alpha < \beta$ such that $B^{\beta}(w) < \inf Q$ and $\sup Q \leq B^{\alpha}(w)$. Then

$$Q \subseteq \left(B^{\beta}(w), B^{\alpha}(w)\right] = \bigcup_{n=\alpha}^{\beta-1} S_n.$$

Therefore,

$$\inf T(Q) \geq \min_{\alpha\leq n<\beta} L_n > -\infty$$

and

$$\sup T(Q) \leq \max_{\alpha\leq n<\beta} U_n < \infty,$$

so T is bounded on Q. Thus (3) implies (6) as required. □

If T is a $\Theta(1)$ solution of the recurrence, then

$$\lim_{x\to\infty} T(x) = 100$$

by Lemma 13.20. However, the family of $\Theta(1)$ solutions does not approach 100 uniformly:

Lemma 13.21. For all $c > 0$ there exists a solution T of the recurrence and an interval $I \subset (10000, \infty)$ with $length(I) > c$ such that T is $\Theta(1)$ and $T(x) = e^x$ for all $x \in I$.

Proof. Let $c > 0$. Observe that

$$\lim_{x\to\infty}\left(x - B(x)\right) = \lim_{x\to\infty}\left(\frac{x}{100} - \sqrt{x}\right) = \infty,$$

so there exists $z > 10000$ such that $z - B(z) > c$. Let $I = (B(z), z)$, so

$$length(I) = z - B(z) > c.$$

Lemma 13.14 implies I is contained in a transversal of the relation $\sim$, so I is independent relative to $\sim$. Corollary 13.10 implies there exists a solution T of the recurrence such that $T(x) = e^x$ for all $x \in I$, so T is bounded on I. Lemma 13.20 implies T is $\Theta(1)$. □

Let T be as in Lemma 13.21, so T is a $\Theta(1)$ solution of the recurrence such that $T(u) \neq 100$ for some $u > 10000$. Lemma 13.19 implies T is unbounded on the bounded interval $(10000,10001)$. We conclude that the condition $x > 10000$ of Lemma 13.20(6) cannot be replaced with the condition $x \geq 10000$.

For future reference. The following proposition is used in Section 19 by a critique of Leighton's Lemma 2.

Lemma 13.22. If $x_0 = 10000$, then

$$\lim_{x \to x_0^+} \Big(x - \big(bx + h(x)\big)\Big) = 0.$$

Proof. Lemma 13.1(3) implies

$$x_0 < B(x) < x$$

for all $x > x_0$, so

$$0 < x - B(x) < x - x_0$$

for all such x. Then

$$0 \leq \liminf_{x \to x_0^+} \big(x - B(x)\big) \leq \limsup_{x \to x_0^+} \big(x - B(x)\big) \leq \lim_{x \to x_0^+} (x - x_0) = 0,$$

so

$$\liminf_{x \to x_0^+} \big(x - B(x)\big) = \limsup_{x \to x_0^+} \big(x - B(x)\big) = 0,$$

i.e.,

$$\lim_{x \to x_0^+} \big(x - B(x)\big) = 0.$$

The proposition follows from $B(x) = bx + h(x)$ for all $x > x_0$. □

14. Satisfaction of Hypothesis by Infinitely Recursive Counterexamples

In this section, we show that the family of recurrences in Section 13 satisfies the hypothesis of Theorem 2 of [Le] with $p = -1$ and $\varepsilon = 0.74$. Observe that $ab^p = 1$ and $ab^q \neq 1$ for all $q \in \boldsymbol{R}\backslash\{p\}$ as required by Theorem 2.

As in Section 13, we let $x_0 \in [686{,}10000]$ and $a = b = 99/100$ and define functions $g, h\colon (0,\infty) \to \boldsymbol{R}$ by $g(x) = 1$ and

$$h(x) = \begin{cases} 0, & \text{for } 1 \leq x \leq x_0 \\ \sqrt{x}, & \text{for } x > x_0. \end{cases}$$

In the notation of [Le], we have $a_1 = a$, $b_1 = b$, $h_1 = h$ and $k = 1$. Condition 1 of Theorem 2 is obviously satisfied: $a > 0$, $b \in (0,1)$, k is a positive integer, the domain of the recurrence is $[1,\infty)$, the function g is non-negative and satisfies Leighton's polynomial-growth condition relative to $\{b\}$ (with $c_1 = c_2 = 1$), and

$$x_0 > 100 = \max(100/99\,, 100) = \max(1/b\,, 1/(1-b)).$$

As explained in Section 13, condition 3 of Leighton's Theorem 2 is satisfied with a caveat: There is a potential ambiguity in the statement of the condition. Some readers may consider the condition to be satisfied only when $x_0 = 10000$.

For all $x \geq x_0$, we have

$$\log^{\varepsilon/2} x \geq \log^{\varepsilon/2} x_0 \geq \log^{0.37} 686 \approx 2.002 > 2,$$

so

$$\frac{1}{2}\left(1 + \frac{1}{\log^{\varepsilon/2} x}\right) < \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4} < 1$$

and

$$2\left(1 - \frac{1}{\log^{\varepsilon/2} x}\right) > 2\left(1 - \frac{1}{2}\right) = 1.$$

Thus conditions 4(c) and 4(d) of Leighton's Theorem 2 are satisfied for all such x.

Satisfaction of condition 2 for our choice of ε is a consequence of Lemma 14.2, whose proof uses the proposition below.

Lemma 14.1. Let δ be a real number, and define $\lambda: (1, \infty) \to \boldsymbol{R}$ by

$$\lambda(t) = \sqrt{t} - \log^{\delta} t.$$

If $x \in (1, \infty)$ such that $\lambda(x) \geq 0$ and $\log x > 2\delta$, then $\lambda'(x) > 0$.

Proof. It follows from

$$\lambda'(x) = \frac{\sqrt{x} - 2\delta \log^{\delta-1} x}{2x},$$

$\log x > 2\delta$, and $\left(\log^{\delta-1} x\right)/(2x) > 0$ that

$$\lambda'(x) > \frac{\sqrt{x} - \log^{\delta} x}{2x} = \frac{\lambda(x)}{2x} \geq 0.$$

□

Lemma 14.2. Let δ be a real number and define $\lambda: (1, \infty) \to \boldsymbol{R}$ by

$$\lambda(t) = \sqrt{t} - \log^{\delta} t.$$

If $c \in (1, \infty)$ such that $\lambda(c) \geq 0$ and $\log c > 2\delta$, then $\lambda|_{[c,\infty)}$ is an increasing function. In particular, $\lambda(x) > 0$ for all $x > c$.

Proof. Let

$$S = \{x > c : \lambda \text{ and } \lambda' \text{ are positive on } (c, x]\},$$

so $S \subseteq (c, \infty)$ and $\inf S \geq c$. Observe that λ and λ' are positive on S.

For all $w \in S$ and all $v \in (c, w]$ we have $v > c$. The functions λ and λ' are positive on $(c, v]$ because $(c, v] \subseteq (c, w]$. Thus $v \in S$ for all such v, i.e., $(c, w] \subseteq S$ for all $w \in S$.

The set S is connected, i.e., S is an interval, because

$$(\alpha, \beta) \subset (c, \beta] \subseteq S.$$

for all $\alpha, \beta \in S$ with $\alpha < \beta$.

Lemma 14.1 implies $\lambda'(c) > 0$. Continuity of λ' implies there exists $d > c$ such that λ' is positive on $(c, d]$. Then λ is positive on $(c, d]$ since $\lambda(c) \geq 0$. Therefore, $d \in S$ (in particular, $S \neq \emptyset$) and $\sup S \geq d$. Furthermore, $(c, d] \subseteq S$, so $\inf S \leq c$. Therefore, $\inf S = c$.

Let $y = \sup S$, so $y > c$. Connectivity of S implies $(c, y) \subseteq S$, so λ and λ' are positive on (c, y). We claim $y = \infty$. Suppose instead that $y < \infty$. Positivity of λ' on (c, y) implies

$$\lambda(y) > \lambda(c) \geq 0.$$

Continuity of λ implies there exists $z > y$ such that λ is positive on $[y, z]$, so λ is positive on

$$(c, z] = (c, y) \cup [y, z].$$

For all $u \in (c, z]$, we have

$$\log u > \log c > 2\delta.$$

Lemma 14.1 implies λ' is positive on $(c, z]$. Thus $z \in S$, which contradicts

$$z > y = \sup S.$$

Therefore, $y = \infty$, i.e., $\sup S = \infty$. The set S is a subinterval of (c, ∞) with $\inf S = c$, so $S = (c, \infty)$. The lemma follows. □

Lemma 14.2 combines with

$$\sqrt{686} - \log^{1+\varepsilon} 686 = \sqrt{686} - \log^{1.74} 686 \approx 0.00673 > 0$$

and

$$\log 686 \approx 6.53 > 3.48 = 2(1 + \varepsilon)$$

to imply

$$\sqrt{t} > \log^{1+\varepsilon} t$$

for all $t \geq 686$. If $x > x_0$, then $x > 686$ and

$$|h(x)| = \sqrt{x} < \sqrt{x} \cdot \frac{\sqrt{x}}{\log^{1+\varepsilon} x} = \frac{x}{\log^{1+\varepsilon} x}.$$

Of course,

$$|h(x_0)| = 0 < \frac{x_0}{\log^{1+\varepsilon} x_0}.$$

Therefore, condition 2 of Theorem 2 is satisfied. It remains to establish compliance with conditions 4(a) and 4(b). We start with a simple observation:

Lemma 14.3. If $u > 0$ and $0 < \alpha < 1$ then

$$(1 + u)^{\alpha} < 1 + \alpha u.$$

Proof. Define $\lambda: \boldsymbol{R} \to \boldsymbol{R}$ by

$$\lambda(x) = 1 + ux - (1 + u)^{x}.$$

Since

$$\lambda(0) = \lambda(1) = 0,$$

there exists $c \in (0,1)$ such that $\lambda'(c) = 0$. The second derivative of λ is

$$\lambda''(x) = -(1+u)^x \log^2(1+u).$$

Positivitiy of u implies λ'' is a negative function, so λ' is a decreasing function. Therefore, $\lambda'|_{(-\infty,c)} > 0$ and $\lambda'|_{(c,\infty)} < 0$, so $\lambda|_{(-\infty,c]}$ is increasing and $\lambda|_{[c,\infty)}$ is decreasing. Then $\lambda(x) > \lambda(0) = 0$ for all $x \in (0,c]$ and $\lambda(x) > \lambda(1) = 0$ for all $x \in [c,1)$. The function λ is positive on $(0,1)$ because $(0,1) = (0,c] \cup [c,1)$. In particular, $\lambda(\alpha) > 0$. □

Conditions 4(a) and 4(b). Let $x \geq x_0$, so $x \geq 686$. Since $p = -1$, conditions 4(a) and 4(b) of Leighton's Theorem 2 are equivalent to the inequalities

$$\left(1 - \frac{1}{b\log^{1+\varepsilon} x}\right)^{-1}\left(1 + \frac{1}{\log^{\varepsilon/2}\left(bx + \frac{x}{\log^{1+\varepsilon} x}\right)}\right) \geq 1 + \frac{1}{\log^{\varepsilon/2} x}$$

and

$$\left(1 + \frac{1}{b\log^{1+\varepsilon} x}\right)^{-1}\left(1 - \frac{1}{\log^{\varepsilon/2}\left(bx + \frac{x}{\log^{1+\varepsilon} x}\right)}\right) \leq 1 - \frac{1}{\log^{\varepsilon/2} x}$$

respectively. Observe that

$$\log x \geq \log 686 > 1,$$

$$b\log^{1+\varepsilon} x \geq b\log^{1+\varepsilon} 686 = \frac{99}{100} \cdot \log^{1.74} 686 > 1,$$

$$bx + \frac{x}{\log^{1+\varepsilon} x} > bx \geq \frac{99 \cdot 686}{100} > e,$$

and

$$\log^{\varepsilon/2}\left(bx + \frac{x}{\log^{1+\varepsilon} x}\right) > \log^{\varepsilon/2}(e) = 1.$$

Let

$$z = \frac{1}{b\log^{1+\varepsilon} x},$$

so $0 < z < 1$, which implies $1 - z > 0$. We conclude from

$$(1-z)(1+z) = 1 - z^2 < 1$$

that

$$(1-z)^{-1} > (1+z) > 1,$$

i.e.,

$$\left(1 - \frac{1}{b\log^{1+\varepsilon} x}\right)^{-1} > 1 + \frac{1}{b\log^{1+\varepsilon} x} > 1.$$

Positivity of z implies $(1+z)^{-1} < 1$, i.e.,

$$\left(1+\frac{1}{b\log^{1+\varepsilon} x}\right)^{-1} < 1.$$

Observe that

$$1+\frac{1}{\log^{\varepsilon/2}\left(bx+\frac{x}{\log^{1+\varepsilon} x}\right)}$$

and

$$1-\frac{1}{\log^{\varepsilon/2}\left(bx+\frac{x}{\log^{1+\varepsilon} x}\right)}$$

are positive. Therefore,

$$\left(1-\frac{1}{b\log^{1+\varepsilon} x}\right)^{-1}\left(1+\frac{1}{\log^{\varepsilon/2}\left(bx+\frac{x}{\log^{1+\varepsilon} x}\right)}\right) > 1+\frac{1}{\log^{\varepsilon/2}\left(bx+\frac{x}{\log^{1+\varepsilon} x}\right)}$$

and

$$\left(1+\frac{1}{b\log^{1+\varepsilon} x}\right)^{-1}\left(1-\frac{1}{\log^{\varepsilon/2}\left(bx+\frac{x}{\log^{1+\varepsilon} x}\right)}\right) < 1-\frac{1}{\log^{\varepsilon/2}\left(bx+\frac{x}{\log^{1+\varepsilon} x}\right)}.$$

Conditions 4(a) and 4(b) are true if

$$b+\frac{1}{\log^{1+\varepsilon} x} \le 1.$$

Therefore, we may assume

$$b+\frac{1}{\log^{1+\varepsilon} x} > 1.$$

Let

$$c = \log\left(b+\frac{1}{\log^{1+\varepsilon} x}\right),$$

so $c > 0$. Since $b < 1$, we have

$$c < \log\left(1+\frac{1}{\log^{1+\varepsilon} x}\right).$$

The function $\lambda: [0,\infty) \to \boldsymbol{R}$ defined by $\lambda(t) = t - \log(1+t)$ is positive on the interval $(0,\infty)$ because $\lambda(0) = 0$ and the derivative $\lambda'(t) = t/(1+t)$ is positive on $(0,\infty)$. Therefore,

$$c < \frac{1}{\log^{1+\varepsilon} x}.$$

Define

$$d = \left(1+\frac{c}{\log x}\right)^{\varepsilon/2}.$$

Positivity of c, ε, and $\log x$ implies $d > 1$. Lemma 14.3 implies

$$d < 1 + \frac{c\varepsilon}{2\log x}$$

since $\varepsilon/2 = 0.37 \in (0,1)$. Observe that

$$\log\left(bx + \frac{x}{\log^{1+\varepsilon} x}\right) = c + \log x.$$

Now

$$\left(1 - \frac{1}{b\log^{1+\varepsilon} x}\right)^{-1}\left(1 + \frac{1}{\log^{\varepsilon/2}\left(bx + \frac{x}{\log^{1+\varepsilon} x}\right)}\right)$$

$$> \left(1 + \frac{1}{b\log^{1+\varepsilon} x}\right)\left(1 + \frac{1}{(c + \log x)^{\varepsilon/2}}\right)$$

$$> 1 + \left(1 + \frac{1}{b\log^{1+\varepsilon} x}\right)\left(\frac{1}{(c + \log x)^{\varepsilon/2}}\right).$$

Condition 4(a) is satisfied if

$$\left(1 + \frac{1}{b\log^{1+\varepsilon} x}\right)\left(\frac{1}{(c + \log x)^{\varepsilon/2}}\right) > \frac{1}{\log^{\varepsilon/2} x},$$

which is equivalent to

$$1 + \frac{1}{b\log^{1+\varepsilon} x} > d.$$

Therefore, condition 4(a) holds if

$$\frac{1}{b\log^{1+\varepsilon} x} > \frac{c\varepsilon}{2\log x}.$$

As stated earlier,

$$\frac{1}{\log^{1+\varepsilon} x} > c.$$

It suffices to show

$$2\log x > b\varepsilon,$$

which follows from $\log x > 1$ and $b\varepsilon = 0.99 \cdot 0.74 < 1$. Therefore, condition 4(a) is satisfied. Condition 4(b) can be written as

$$\left(1 + \frac{1}{b\log^{1+\varepsilon} x}\right)^{-1}\left(1 - \frac{1}{d\log^{\varepsilon/2} x}\right) \le 1 - \frac{1}{\log^{\varepsilon/2} x},$$

which is equivalent to

$$1 + \frac{d-1}{d(\log^{\varepsilon/2} x - 1)} = \frac{d \log^{\varepsilon/2} x - 1}{d(\log^{\varepsilon/2} x - 1)} \leq 1 + \frac{1}{b \log^{1+\varepsilon} x},$$

i.e.,

$$b(d-1) \log^{1+\varepsilon} x \ \leq \ d\big(\log^{\varepsilon/2} x - 1\big).$$

Observe that

$$b(d-1) \log^{1+\varepsilon} x < \frac{bc\varepsilon}{2 \log x} \log^{1+\varepsilon} x = \frac{bc\varepsilon \log^{\varepsilon} x}{2} < \frac{1}{\log^{1+\varepsilon} x} \cdot \frac{b\varepsilon \log^{\varepsilon} x}{2}$$

$$= \frac{b\varepsilon}{2 \log x} \leq \frac{b\varepsilon}{2 \log 686} = \frac{0.99 \cdot 0.74}{2 \log 686} \approx 0.056$$

and

$$d\big(\log^{\varepsilon/2} x - 1\big) > \log^{\varepsilon/2} x - 1 \geq \log^{\varepsilon/2} 686 - 1 = \log^{0.37} 686 - 1 \approx 1.002.$$

Therefore, condition 4(b) holds and the hypothesis of Theorem 2 is satisfied.

15. A Finitely Recursive Counterexample to Leighton's Theorem 2

Section 13 describes a family of infinitely recursive counterexamples to Theorem 2 of [Le]. Each member of the family has a solution that agrees with the Akra-Bazzi formula but also has infinitely many solutions that differ wildly from the Akra-Bazzi formula.

In this section, we define a related (proper) divide-and-conquer recurrence that also satisfies the hypothesis of Theorem 2. It is finitely recursive and hence has a unique solution. However, the solution does not conform to the Akra-Bazzi formula.

Let $x_0 = 10000$ and define b, B, and $\sim$ as in Section 13, i.e., $b = 99/100$, the function $B: (0, \infty) \to (0, \infty)$ is defined by $B(x) = bx + \sqrt{x}$, and $\sim$ is the equivalence relation on (x_0, ∞) with $\gamma \sim \delta$ when there exists an integer i with $B^i(\gamma) = \delta$.

Lemma 13.3 implies B is a bijection of $(0, \infty)$ onto itself, so each integral power of B is defined and is a bijection from $(0, \infty)$ onto itself. Here as in Section 13, powers of B represent composition of functions. Lemma 13.4(3) implies (x_0, ∞) is invariant under each integral power of B.

Lemma 13.1(3) implies

$$x_0 < B(x_0 + 1) < x_0 + 1.$$

Define a half-open interval

$$Y = (B(x_0 + 1), x_0 + 1].$$

Lemma 13.14 implies Y is a transversal of the equivalence relation $\sim$. By Lemma 13.7(3), for each $x \in (x_0, \infty)$ there exists a unique corresponding integer α with $B^\alpha(x) \in Y$. Define $n: (x_0, \infty) \to \boldsymbol{Z}$ by $B^{n(x)}(x) \in Y$. The function $x \mapsto B^{n(x)}(x)$ on (x_0, ∞) is constant on each equivalence class, i.e.,

$$B^{n(x_1)}(x_1) = B^{n(x_2)}(x_2)$$

when $x_1, x_2 \in (x_0, \infty)$ with $x_1 \sim x_2$. Observe that $n(x) = 0$ if and only if $x \in Y$. When $x > x_0 + 1$, we have

$$B^{n(x)}(x) < x = B^0(x),$$

so $n(x) > 0$ by Lemma 13.5. When $x \leq B(x_0 + 1)$, we have

$$B^{n(x)}(x) > x = B^0(x),$$

so $n(x) < 0$ by Lemma 13.5. B-invariance of (x_0, ∞) implies is $B(x) \in domain(n)$ for all $x \in (x_0, \infty)$. Of course, $n(x) = n\big(B(x)\big) + 1$ for all such x because

$$B^{n(B(x))+1}(x) = B^{n(B(x))}\big(B(x)\big) \in Y.$$

For each positive integer j, define

$$t_j = B(x_0 + 1) + \frac{(x_0 + 1) - B(x_0 + 1)}{j}.$$

Then $t_1, t_2, t_3, \ldots$ is a decreasing sequence in Y with $t_1 = x_0 + 1$ and

$$\lim_{j \to \infty} t_j = B(x_0 + 1).$$

For each positive integer j, define the half-open interval

$$Y_j = \big(t_{j+1}, t_j\big].$$

Observe that $Y_1, Y_2, Y_3, \ldots$ are disjoint non-empty sets and

$$Y = \bigcup_{j=1}^{\infty} Y_j.$$

There exists a surjection $\lambda: Y \to \boldsymbol{Z}^+$ with $y \in Y_{\lambda(y)}$ for all $y \in Y$. Define a non-negative integer-valued function $d: [1, \infty) \to \boldsymbol{N}$ by

$$d(x) = \begin{cases} 0, & \text{for } x \leq x_0 \\ \max\left(n(x) + \lambda\left(B^{n(x)}(x)\right), 1\right), & \text{for } x > x_0, \end{cases}$$

so $d(y) = \lambda(y)$ for all $y \in Y$. Observe that $\boldsymbol{Z}^+ = \lambda(Y) = d(Y) \subseteq range(d)$, which implies d is unbounded on Y and d is a surjection onto $\boldsymbol{N}$. The function d is positive on (x_0, ∞). When $d(x) > 1$, we have $x > x_0$ and

$$d(x) = n(x) + \lambda\left(B^{n(x)}(x)\right);$$

of course, $x \sim B(x)$, so

$$B^{n(B(x))}\big(B(x)\big) = B^{n(x)}(x),$$

which combines with $n\big(B(x)\big) = n(x) - 1$ to imply

$$d\big(B(x)\big) = \max\left(n(x) - 1 + \lambda\left(B^{n(x)}(x)\right), 1\right) = \max(d(x) - 1, 1) = d(x) - 1.$$

When $d(x) = 1$, we have $x > x_0$ and $n(x) \le 0$, so $x \in (x_0, x_0 + 1]$, which implies

$$bx \in (bx_0, bx_0 + b] \subset (bx_0, x_0] \subset [1, x_0],$$

so $d(bx) = 0$. Since (x_0, ∞) is B-invariant and $(bx_0, \infty) = (bx_0, x_0] \cup (x_0, \infty)$, there is a function $r: (x_0, \infty) \to (bx_0, \infty)$ defined by

$$r(x) = \begin{cases} bx, & \text{for } d(x) = 1 \\ B(x), & \text{for } d(x) > 1. \end{cases}$$

Observe that $range(r) \subset [1, \infty)$. Lemma 13.1(3) and $b < 1$ imply $r(x) < x$ for all $x \in domain(r)$.

When $d(x) = 1$, we have $r(x) \in [1, x_0]$ and $d\big(r(x)\big) = 0 = d(x) - 1$. When $d(x) > 1$, we have

$$d\big(r(x)\big) = d\big(B(x)\big) = d(x) - 1,$$

i.e., d satisfies the recurrence

$$d(x) = \begin{cases} 0, & \text{for } x \in [1, x_0] \\ d\big(r(x)\big) + 1, & \text{for } x > x_0. \end{cases}$$

Since $range(d) = \boldsymbol{N}$, the recurrence above satisfied by d must be finitely recursive. Lemma 8.2 implies d is its unique solution.

Define $g: [b, \infty) \to \boldsymbol{R}$ by $g(x) = 1$, and define $h: [1, \infty) \to \boldsymbol{R}$ by

$$h(x) = \begin{cases} 0, & \text{when } d(x) \le 1 \\ \sqrt{x}, & \text{when } d(x) > 1, \end{cases}$$

so h is zero on $[1, x_0]$, and $r(x) = bx + h(x)$ for all $x \in (x_0, \infty)$. Let $D = [1, \infty)$, $I = (x_0, \infty)$, and $a = 1$. Define the constant function $f: [1, x_0] \to \boldsymbol{R}$ by $f(x) = 1$. Then

$$(D, I, a, b, f, g|_I, h|_I)$$

is a divide-and-conquer recurrence, and $d(x)$ is the depth-of-recursion at x relative to $D - I$ for all $x \in D$. In particular, recursion is finite. Lemma 8.2 implies the recurrence has a unique solution $T: D \to \boldsymbol{R}$, which satisfies

$$T(x) = \begin{cases} 1, & \text{for } x \in [1, x_0] \\ T\big(r(x)\big) + 1, & \text{for } x > x_0. \end{cases}$$

Define

$$S = \{m \in \boldsymbol{N} : T(x) = d(x) + 1 \text{ for all } x \in d^{-1}(\{m\})\ \}.$$

Here $d^{-1}(\{m\})$ is the preimage of $\{m\}$ under d. Observe that $d^{-1}(\{0\}) = [1, x_0]$, and $T(v) = 1 = d(v) + 1$ for all $v \in [1, x_0]$, so $0 \in S$. Suppose $m \in S$, so $m + 1 > 0$. If $w \in d^{-1}(\{m + 1\})$, then $w > x_0$, so

$$d\big(r(w)\big) = d(w) - 1 = m.$$

Then $T\big(r(w)\big) = m + 1$, so

$$T(w) = T\big(r(w)\big) + 1 = m + 2 = d(w) + 1.$$

Therefore, $m + 1 \in S$. By induction, $S = \boldsymbol{N}$, so $T(x) = d(x) + 1$ for all

$$x \in \bigcup_{m=0}^{\infty} d^{-1}(\{m\}) = d^{-1}(\boldsymbol{N}) = [1, \infty),$$

i.e., $T(x) = d(x) + 1$ for all $x \in D$.

We observe that our recurrence for T has unbounded depth of recursion on Y because d is unbounded on Y. Furthermore, the unique solution T is unbounded on Y because $T - d$ is constant. The set Y is bounded, so the recurrence does not satisfy the bounded depth condition, and the solution T is not locally $\Theta(1)$.

Theorem 2 of [Le] applies to recurrences of the form

$$\tau(x) = \begin{cases} \Theta(1), & \text{for } 1 \le x \le x_0 \\ \displaystyle\sum_{i=1}^{k} a_i\, \tau\big(b_i x + h_i(x)\big) + g(x), & \text{for } x > x_0 \end{cases}$$

that satisfy four conditions. We will show that our recurrence for T satisfies the hypothesis of Leighton's Theorem 2 with $a_1 = a$, $b_1 = b$, $k = 1$, $h_1 = h$, our choices of x_0 and g, and

$$\varepsilon = \frac{\log\sqrt{x_0}}{\log\log x_0} - 1 = \frac{\log 100}{\log\log 10000} - 1 \approx 1.074.$$

Observe that T is $\Theta(1)$ on $[1, x_0]$ and

$$T(x) = \sum_{i=1}^{k} a_i\, T\big(b_i x + h_i(x)\big) + g(x)$$

for all $x > x_0$ as required. Condition 1 of Theorem 2 is satisfied because the domain of the recurrence is $[1, \infty)$,

$$x_0 = 10000 > \max(100/99\,, 100) = \max(1/b\,, 1/(1-b)),$$

$a > 0$, $b \in (0,1)$, $k = 1$, and g is a non-negative function that satisfies Leighton's polynomial-growth condition relative to $\{b\}$ with $c_1 = c_2 = 1$. Observe that

$$bx + h(x) \in [b, \infty) = domain(g)$$

for all $x \geq 1$ because $b > 0$ and h is non-negative. Since g is constant, we conclude that condition 3 of Theorem 2 is satisfied with $c_1 = c_2 = 1$.

Let $x \geq x_0$. Observe that

$$\log(\log^{1+\varepsilon} x_0) = (1+\varepsilon)(\log\log x_0) = \log\sqrt{x_0},$$

so

$$\log^{1+\varepsilon} x_0 = \sqrt{x_0} = 100.$$

Lemma 14.2 combines with

$$\sqrt{x_0} - \log^{1+\varepsilon} x_0 = 0$$

and

$$\log x_0 = \log 10000 \approx 9.2 > 4.15 \approx 2(1+\varepsilon)$$

to imply

$$\sqrt{x} \geq \log^{1+\varepsilon} x.$$

Of course, $\log x > 0$, so

$$|h(x)| \;\leq\; \sqrt{x} \;\leq\; \sqrt{x} \cdot \frac{\sqrt{x}}{\log^{1+\varepsilon} x} = \frac{x}{\log^{1+\varepsilon} x}$$

as required by condition 2 of Theorem 2.

Theorem 2 defines p to be the solution of

$$\sum_{i=1}^{k} a_i\, b_i^p = 1,$$

i.e., $b^p = 1$, so $p = 0$. Observe that

$$b\log^{1+\varepsilon} x \geq b\log^{1+\varepsilon} x_0 = 99,$$

which implies

$$1 - \frac{1}{b\log^{1+\varepsilon} x} > 0$$

and

$$1 + \frac{1}{b\log^{1+\varepsilon} x} > 0,$$

so

$$\left(1-\frac{1}{b\log^{1+\varepsilon} x}\right)^p = \left(1+\frac{1}{b\log^{1+\varepsilon} x}\right)^p = 1.$$

In particular, the fractions and the pth powers are defined. Now conditions 4(a) and 4(b) are equivalent to

$$1+\frac{1}{\log^{\varepsilon/2}\left(bx+\frac{x}{\log^{1+\varepsilon} x}\right)} \geq 1+\frac{1}{\log^{\varepsilon/2} x}$$

and

$$1-\frac{1}{\log^{\varepsilon/2}\left(bx+\frac{x}{\log^{1+\varepsilon} x}\right)} \leq 1-\frac{1}{\log^{\varepsilon/2} x}$$

respectively, which are equivalent to

$$b+\frac{1}{\log^{1+\varepsilon} x} \leq 1$$

since $\varepsilon > 0$. The various denominators are defined and positive because $x \geq x_0 > 0$,

$$bx \geq bx_0 > 1,$$

and

$$\log x \geq \log x_0 > 0.$$

In particular, the denominators are non-zero and the fractions are defined. Observe that $1+\varepsilon > \varepsilon > 0$, so

$$b+\frac{1}{\log^{1+\varepsilon} x} \leq b+\frac{1}{\log^{1+\varepsilon} x_0} = \frac{99}{100}+\frac{1}{100} = 1,$$

i.e., conditions 4(a) and 4(b) are satisfied. The inequalities $\log x_0 > 1$ and $\varepsilon > 1$ imply

$$\log^{\varepsilon/2} x \geq \log^{\varepsilon/2} x_0 > \sqrt{\log x_0} \approx 3 > 2,$$

so

$$\frac{1}{2}\left(1+\frac{1}{\log^{\varepsilon/2} x}\right) < 1$$

and

$$2\left(1-\frac{1}{\log^{\varepsilon/2} x}\right) > 1,$$

which are strict versions of conditions 4(c) and 4(d). We conclude that the recurrence satisfies the hypothesis of Leighton's Theorem 2.

Suppose T satisfies the Akra-Bazzi formula, which says (with 0 substituted for p) that

$$T(x) = \Theta\left(x^0\left(1 + \int_1^x \frac{1}{u^{0+1}} du\right)\right) = \Theta(\log x).$$

Then there exists $z \geq 1$ such that T is bounded on each bounded subset of (z, ∞). Lemma 13.6(3) implies the existence of a negative integer q such that

$$B^{q+1}(x_0 + 1) > z.$$

Lemmas 13.4(1) and continuity of B imply

$$B^q(Y) = (B^{q+1}(x_0 + 1), B^q(x_0 + 1)],$$

so $B^q(Y)$ is a bounded subset of (z, ∞), which implies T is bounded on $B^q(Y)$. Then d is bounded on $B^q(Y)$ because $T - d$ is constant. However, for each positive integer j, we have $B^q(t_j) \in B^q(Y)$,

$$n\left(B^q(t_j)\right) = -q > 0,$$

and

$$\lambda\left(B^{-q}\left(B^q(t_j)\right)\right) = \lambda(t_j) = j,$$

so

$$d\left(B^q(t_j)\right) = j - q > j.$$

Therefore, d is unbounded on $B^q(Y)$. We conclude that T violates the Akra-Bazzi formula.

For future reference. The following proposition is used in Section 19 by a critique of Leighton's Lemma 2.

Lemma 15.1. The divide-and-conquer recurrence defined in this section satisfies

$$\inf_{x_0 < bx + h(x) < x < x_0 + 1} \left(x - \left(bx + h(x)\right)\right) = 0.$$

Proof. Let μ be the infimum defined above, so $\mu \geq 0$. Let $\delta > 0$. Lemmas 13.1(3), 13.4, and 13.6(1) imply the existence of a positive integer m with the property that

$$x_0 < B(\zeta) < \zeta < B^m(x_0 + 1) < \min(x_0 + \delta, x_0 + 1)$$

where $\zeta = B^m(t_{m+2})$. (Recall that $x_0 < t_{m+2} < t_1 = x_0 + 1$.) Observe that

$$\zeta - B(\zeta) < \delta.$$

We conclude from

$$B^{-m}(\zeta) = t_{m+2} \in (t_{m+3}, t_{m+2}] = Y_{m+2} \subset Y$$

that $n(\zeta) = -m$ and $\lambda\left(B^{-m}(\zeta)\right) = m + 2$, so $d(\zeta) = 2$, which implies $r(\zeta) = B(\zeta)$, i.e.,

$$B(\zeta) = b\zeta + h(\zeta),$$

so

$$x_0 < b\zeta + h(\zeta) < \zeta < x_0 + 1$$

and

$$\zeta - \left(b\zeta + h(\zeta)\right) < \delta.$$

Then $\mu < \delta$ for all $\delta > 0$, so $\mu \leq 0$, which combines with $\mu \geq 0$ to imply $\mu = 0$. □

16. Base Case of the Induction

Leighton's Theorem 2 claims all solutions $T: [1, \infty) \rightarrow \boldsymbol{R}$ to certain recurrences of the form

$$T(x) = \begin{cases} \Theta(1), & \text{for } x \in [1, x_0] \\ \displaystyle\sum_{i=1}^{k} a_i T\big(b_i x + h_i(x)\big) + g(x), & \text{for } x > x_0 \end{cases}$$

satisfy the Akra-Bassi formula, $T(x) = \Theta\big(\mathrm{Z}(x)\big)$ where the function $Z: [1, \infty) \rightarrow \boldsymbol{R}$ is defined by

$$Z(x) = x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du\right).$$

Here $x_0, k, a_1, \ldots, a_k, b_1, \ldots, b_k, g, h_1, \ldots, h_k,$ and p are as in [Le]. In particular, $x_0 > 1$.

The claimed inductive proof in [Le] asserts the existence of positive real numbers c_5 and c_6 that satisfy

$$c_5 \left(1 + \frac{1}{\log^{\varepsilon/2} x}\right) \mathrm{Z}(x) \leq T(x) \leq c_6 \left(1 - \frac{1}{\log^{\varepsilon/2} x}\right) \mathrm{Z}(x)$$

for all $x \in (x_0, \infty)$ where $\varepsilon > 0$ satisfies conditions 2 and 4 of Theorem 2. However, the base case of Leighton's inductive argument consists of the inequalities above for all $x \in [1, x_0]$, even though $[1, x_0]$ and (x_0, ∞) are disjoint. When $x = 1$, the inequalities involve division by zero and are equivalent to

$$c_5 \left(1 + \frac{1}{0}\right) \leq T(1) \leq c_6 \left(1 - \frac{1}{0}\right).$$

The only plausible interpretation (in context) is the obviously false chain of inequalities

$$+\infty \leq T(1) \leq -\infty.$$

An implausible interpretation is the trivial chain of inequalities

$$-\infty \leq T(1) \leq +\infty.$$

Fortunately, usage of the base case by the inductive step of the argument only requires that the asserted inequalities hold for all $x \in S$ where

$$S = \{x \leq x_0 : x = b_i t + h_i(t) \text{ for some } t > x_0 \text{ and some } i \in \{1, \ldots, k\}\}.$$

The inductive step requires that $1/\log^{\varepsilon/2}(u)$ is a real number for all $u \in S$. In particular, $1 \notin S$ regardless of how we interpret $1/0$. An obvious attempt to save the argument of [Le] requires that $1 \notin S$. In Section 19 we exhibit a recurrence that satisfies the hypothesis of Theorem 2, but has $1 \in S$ and $\inf S < 1$. However, the intended domain of T in [Le] is evidently $[1, \infty)$. We consider the recurrence described in Theorem 2 to have no solution when $\inf S < 1$. For the remainder of this section, we assume $1 \notin S$ and $\inf S \geq 1$, so $S \subseteq (1, x_0]$. (Indeed, we shall soon strengthen the restrictions on S.) In addition to enforcing consistency with the recurrence's domain and avoiding claims of finite quotients with denominators that are zero, the assumption also eliminates consideration of special cases in the interpretation of $\log^{\varepsilon/2} x$ for $x < 1$. Arbitrary real powers of negative numbers and logarithms of non-positive numbers are of course problematic.

By hypothesis, $T|_{[1,x_0]} = \Theta(1)$, which implies $T|_S = \Theta(1)$. The function g is non-negative and satisfies Leighton's polynomial-growth condition relative to the set $\{b_i : 1 \leq i \leq k\}$. Let

$$b_{min} = \min_{1 \leq i \leq k} b_i,$$

so $domain(g)$ contains $[b_{min}, \infty)$. Corollary 2.17 implies g has polynomial growth on the interval $[b_{min}, \infty)$, which contains $[1, \infty)$. Lemma 2.2(2) implies g has polynomial growth on $[1, \infty)$. Leighton implicitly assumes that g is locally Riemann integrable on $[1, \infty)$, so g is tame on $[1, \infty)$. Lemma 10.6 implies Z is locally $\Theta(1)$, so $Z|_S = \Theta(1)$.

There exist S-compatible candidates for c_5 and c_6 if and only if

$$\sup_{x \in S} \left(1 + \frac{1}{\log^{\varepsilon/2} x}\right) < \infty$$

and

$$\inf_{x \in S} \left(1 - \frac{1}{\log^{\varepsilon/2} x}\right) > 0,$$

respectively, i.e., $\inf S > 1$ and the more stringent $\inf S > e$, respectively. We conclude that a suitable version of the base case of Leighton's inductive argument is true if and only if $\inf S > e$. In later sections, we provide a replacement for Theorem 2 with a modified proof that does not require any such restrictions on S.

Failure of restriction to *S* of base case of induction does not imply conclusion of Theorem 2 is false. Let $x_0 = 2e$ and $a = b = 1/2$. Let g and h be the identically zero functions on $[b, \infty)$ and $[1, \infty)$ respectively. We claim the divide-and-conquer recurrence

$$T(x) = \begin{cases} 1, & \text{for } x \in [1, x_0] \\ aT\big(bx + h(x)\big) + g(x), & \text{for } x > x_0, \end{cases}$$

i.e.,

$$T(x) = \begin{cases} 1, & \text{for } x \in [1, 2e] \\ \frac{1}{2}T\left(\frac{x}{2}\right), & \text{for } x > 2e, \end{cases}$$

satisfies the hypothesis of Leighton's Theorem 2 with $p = -1$ and $\varepsilon = 3$. The base case of the recurrence is obviously $\Theta(1)$. Of course, $a_1 = a$, $b_1 = b$, $h_1 = h$, and $k = 1$ in the language of [Le]. Since $a = b \neq 0$, we have $ab^p = aa^{-1} = 1$ as required.

Condition 1 of Theorem 2 is satisfied because $[1, \infty)$ is the domain of the recurrence,

$$x_0 > 2 = 1/b = 1/(1 - b),$$

$a > 0$, $b \in (0,1)$, $k = 1$, and g is a non-negative function that satisfies Leighton's polynomial-growth condition relative to $\{b\}$ with $c_1 = c_2 = 1$. Condition 2 is satisfied because h is identically zero and $x_0 > 1$. Furthermore,

$$[bx + h(x), x] = [x/2\,, x] \subset [1/2, \infty) = domain(g)$$

for all $x \geq 1$. Then condition 3 is satisfied with $c_1 = c_2 = 1$ because g is identically zero. Conditions 4(c) and 4(d) follow from

$$\log^{\varepsilon/2} x \geq \log^{\varepsilon/2} x_0 = \log^{3/2}(2e) \approx 2.2 > 2$$

for all $x \geq x_0$. Observe that

$$\log^{1+\varepsilon} x \geq \log^{1+\varepsilon} x_0 = \log^4(2e) \approx 8$$

for all such x, so

$$b \log^{1+\varepsilon} x > 1,$$

$$bx + \frac{x}{\log^{1+\varepsilon} x} > bx \geq bx_0 = e,$$

and

$$b + \frac{1}{\log^{1+\varepsilon} x} < 1.$$

As in section 14, inequalities above combine with $p = -1$ to imply conditions 4(a) and 4(b) are satisfied. Therefore, the hypothesis of Theorem 2 is satisfied. Furthermore, the recurrence is finitely recursive. Lemma 8.2 implies there exists a unique solution T. Theorem 2 correctly predicts that

$$T(x) = \Theta\left(\frac{1}{x}\right).$$

However,

$$\inf_{x>x_0}\big(bx + h(x)\big) = e.$$

Therefore, the restriction of the base case of Leighton's induction to the set S is unsatisfied.

Our claim about the asymptotic behavior of the solution T is easily verified. Let d be the depth-of-recursion function for the recurrence, and define

$$A = \{n \in \boldsymbol{Z}^+ : \ xT(x) \in (x_0/2, x_0] \text{ for all } x > x_0 \text{ with } d(x) = n\}.$$

If $u > x_0$ such that $d(u) = 1$, then $u/2 \in [1, x_0]$, so $T(u/2) = 1$ and

$$uT(u) = \frac{u}{2}T\left(\frac{u}{2}\right) = \frac{u}{2} \in (x_0/2, x_0].$$

Therefore $1 \in A$. Now suppose $n \in A$, so $n \geq 1$. Given $w > x_0$ with $d(w) = n + 1$, we have $d(w/2) = n$, so $w/2 > x_0$ and

$$wT(w) = \frac{w}{2}T\left(\frac{w}{2}\right) \in (x_0/2, x_0].$$

We conclude that $n + 1 \in A$. By induction, $A = \boldsymbol{Z}^+$, which combines with finite recursion to imply

$$xT(x) \in (x_0/2, x_0]$$

for all $x > x_0$, i.e.,

$$\frac{e}{x} < T(x) \leq \frac{2e}{x}$$

for all such x. (Indeed, the inequalities are satisfied for all $x > x_0/2$, i.e., $x > e$.) Therefore,

$$T(x) = \Theta\left(\frac{1}{x}\right).$$

as claimed.

Main counterexamples do not involve failure of restricted base case. The counterexamples in Sections 13 and 15 to Theorem 2 cannot be explained by a failure of the induction's base case. They satisfy

$$\inf_{x>x_0}\big(bx + h(x)\big) = 10000 > e$$

and

$$\inf_{x>x_0}\left(bx + h(x)\right) = 9900 > e,$$

respectively (for their choices of b, x_0, and h).

17. Example of Akra-Bazzi Solution Unbounded on $(x_0, x_0 + 1)$

In this section, we define a divide-and-conquer recurrence of the form

$$T(x) = \begin{cases} 1, & \text{for } x \in [1, x_0] \\ T\big(bx + h(x)\big) + 1, & \text{for } x > x_0 \end{cases}$$

that satisfies the hypothesis of Theorem 2 with $x_0 = 10000$ and $p = 0$. As we shall see, the recurrence is finitely recursive, so Lemma 8.2 implies the existence of a unique solution T. Theorem 2 of [Le] predicts that

$$T(x) = \Theta(\log x).$$

We will show that the prediction is correct, but T is unbounded above on the bounded interval $(x_0, x_0 + 1)$. In the context of our recurrence, the claimed proof of Theorem 2 asserts the existence of a positive real number c_6 that satisfies

$$T(x) \le c_6 \left(1 - \frac{1}{\log^{\varepsilon/2} x}\right)(1 + \log x)$$

for all $x > x_0$ where $\varepsilon > 0$ satisfies conditions 2 and 4 of Theorem 2. The assertion implies $c_6(1 + \log(x_0 + 1))$ is an upper bound for T on $(x_0, x_0 + 1)$, which is a contradiction. Therefore, the predicted c_6 does not exist, and the inductive hypothesis is unsatisfied by this example. However, there is no failure of the restricted form of the base case of the induction as described in Section 16. The failure is with the inductive step.

Let x_0, b, Y, d, r, ε, and p be as in Section 15. Define $s: (x_0, \infty) \to [1, \infty)$ by

$$s(x) = \begin{cases} r(x), & \text{for } x \in (x_0, x_0 + 1] \\ \min(bx, x_0), & \text{for } x \in (x_0 + 1, (x_0 + 1)/b] \\ bx, & \text{for } x > (x_0 + 1)/b\,. \end{cases}$$

As in Section 15, define a constant function $g: [b, \infty) \to \boldsymbol{R}$ by $g(x) = 1$. Define $h: [1, \infty) \to \boldsymbol{R}$ by

$$h(x) = \begin{cases} 0, & \text{for } x \in [1, x_0] \\ s(x) - bx, & \text{for } x > x_0. \end{cases}$$

Observe that

$$bx + h(x) = s(x) \in [1, x)$$

for all $x > x_0$. We claim the divide-and-conquer recurrence

$$T(x) = \begin{cases} 1, & \text{for } x \in [1, x_0] \\ T\big(bx + h(x)\big) + g(x), & \text{for } x > x_0, \end{cases}$$

i.e.,

$$T(x) = \begin{cases} 1, & \text{for } x \in [1, x_0] \\ T\big(s(x)\big) + 1, & \text{for } x > x_0, \end{cases}$$

satisfies the hypothesis of Theorem 2. The base case is certainly $\Theta(1)$ as required. The recurrence above is derived from the recurrence in Section 15 and has the same Akra-Bazzi exponent, $p = 0$. Satisfaction of conditions 1 and 4 of Theorem 2 is inherited from the recurrence in Section 15.

Observe that

$$[bx + h(x), x] = [s(x), x] \subset [1, \infty) \subset [b, \infty)$$

for all $x > x_0$, and

$$[bx + h(x), x] = [bx, x] \subset [b, \infty)$$

for all $x \in [1, x_0]$, so

$$[bx + h(x), x] \subset [b, \infty) = domain(g)$$

for all $x \geq 1$. Since g is a constant function, we conclude that condition 3 of Theorem 2 is satisfied.

Satisfaction of condition 2 for $x \in [x_0, x_0 + 1]$ is also inherited from the recurrence in Section 15. Since $h(x) = 0$ when $x \in (x_0 + 1, x_0/b]$ or $x > (x_0 + 1)/b$, we need only verify condition 2 for all

$$x \in (x_0/b, (x_0 + 1)/b].$$

For all such x, we also have

$$x \in (x_0 + 1, (x_0 + 1)/b],$$

so

$$s(x) = \min(bx, x_0) = x_0$$

and

$$|h(x)| = bx - x_0 \leq 1.$$

Define $L: (1, \infty) \to \boldsymbol{R}$ by

$$L(t) = \frac{t}{\log^{1+\varepsilon} t},$$

so the derivative of L is

$$L'(t) = \frac{\log t - 1 - \varepsilon}{\log^{2+\varepsilon} t},$$

which is positive on $(e^{1+\varepsilon}, \infty)$. By definition of ε, we have

$$e^{1+\varepsilon} = e^{\frac{\log 100}{\log\log 10000}} \approx 7.96,$$

so L is increasing on $(8, \infty)$. Recall from Section 15 that $\log^{1+\varepsilon} x_0 = 100$, so

$$L(x_0) = \frac{x_0}{100} = 100,$$

which implies

$$L(x) > 100$$

for all $x > x_0$. Recall that $|h(x)| \leq 1$ for all

$$x \in (x_0/b, (x_0 + 1)/b],$$

so $|h(x)| < L(x)$ for all such x. We conclude that condition 2 of Leighton's Theorem 2 is satisfied, which implies the entire hypothesis of Theorem 2 is satisfied. As previously mentioned, Theorem 2 predicts $T(x) = \Theta(\log x)$.

Define $w: (x_0 + 1, \infty) \to \boldsymbol{Z}^+$ by

$$w(x) = \text{ceiling}\big(\log_{1/b}(x/(x_0 + 1))\big).$$

Define $d^*: [1, \infty) \to \boldsymbol{N}$ by

$$d^*(x) = \begin{cases} d(x), & \text{for } x \in [1, x_0 + 1] \\ w(x), & \text{for } x > x_0 + 1, \end{cases}$$

so $d^*(x) = 0$ for all $x \in [1, x_0]$. For all $x \in (x_0, x_0 + 1]$, we have

$$r(x) \in (bx_0, x) \subset [1, x_0 + 1],$$

so

$$d^*(x) = d(x) = d\big(r(x)\big) + 1 = d^*\big(r(x)\big) + 1 = d^*\big(s(x)\big) + 1.$$

If $x \in (x_0 + 1, (x_0 + 1)/b]$, then $s(x) \in [1, x_0]$ and

$$d^*(x) = w(x) = 1 = d\big(s(x)\big) + 1 = d^*\big(s(x)\big) + 1.$$

If $x > (x_0 + 1)/b$, then

$$s(x) = bx > x_0 + 1$$

and

$$d^*(x) = w(x) = w(bx) + 1 = w\big(s(x)\big) + 1 = d^*\big(s(x)\big) + 1.$$

We conclude that d^* satisfies the recurrence

$$d^*(x) = \begin{cases} 0, & \text{for } x \in [1, x_0] \\ d^*(s(x)) + 1, & \text{for } x > x_0. \end{cases}$$

The recurrence satisfied by d^* is finitely recursive because $range(d^*) \subseteq \boldsymbol{N}$ (indeed, a simple argument shows equality), so d^* is its unique solution by Lemma 8.2. Therefore, $d^*(x)$ is the depth of recursion (for the recurrence satisfied by T) at x for all $x \in [1, \infty)$. In particular, $w(x)$ is the depth of recursion at x for all $x > x_0 + 1$. Observe that d^* is positive on (x_0, ∞).

The function d^* is integer-valued, so our main recurrence is finitely recursive and therefore has a unique solution, T, by Lemma 8.2. A simple inductive argument on $d^*(x)$ shows that $T(x) = d^*(x) + 1$ for all $x \in [1, \infty)$. Therefore, $T(x) = w(x) + 1$ for all $x > x_0 + 1$. For all such x, we have

$$w(x) = \text{ceiling}\left(\frac{\log x - \log(x_0 + 1)}{|\log b|}\right) < \frac{\log x}{|\log b|} - 915.43$$

because $0 < \mathrm{b} < 1$ and

$$\frac{\log(x_0 + 1)}{|\log b|} \approx 916.431 > 916.43,$$

so

$$T(x) < \frac{\log x}{|\log b|}.$$

(The sharper upper bound for $w(x)$ will be used in Section 18.) If $x > (x_0 + 1)^2$ (so also $x > x_0 + 1$), then $\log(x_0 + 1) < (\log x)/2$, so

$$T(x) > w(x) \geq ceiling\left(\frac{\log x}{2|\log b|}\right) \geq \frac{\log x}{2|\log b|}.$$

Therefore,

$$\frac{\log x}{2|\log b|} < T(x) < \frac{\log x}{|\log b|}$$

for all $x > (x_0 + 1)^2$. We conclude that $T(x) = \Theta(\log x)$, as predicted by Theorem 2. However, d is unbounded on Y, which is contained in $(x_0, x_0 + 1]$. The functions d and d^* agree on Y, so d^* is unbounded on Y, which implies T is unbounded on Y. Therefore, T is unbounded on $(x_0, x_0 + 1]$, which implies T is unbounded on $(x_0, x_0 + 1)$ as claimed. The appropriately restricted form of the base case of the induction (as described in Section 16) is satisfied because

$$\inf_{x > x_0}\big(bx + h(x)\big) = \inf_{x > x_0}\big(s(x)\big) = \inf_{x \in (x_0, x_0 + 1]}\big(r(x)\big) = 9900 > e.$$

18. Example of Akra-Bazzi Solution With $\inf T(x) = 0$ on $(x_0, x_0 + 1)$

Let x_0, b, Y, d, r, and ε be as in Sections 15 and 17, and let s, h, w, and d^* be as in Section 17. Define $a = b$, i.e., $a = 99/100$ and let $g: [b, \infty) \to \boldsymbol{R}$ be identically zero. We will show that the divide-and-conquer recurrence

$$T(x) = \begin{cases} 1, & \text{for } x \in [1, x_0] \\ aT\big(s(x)\big), & \text{for } x > x_0 \end{cases}$$

satisfies the hypothesis of Theorem 2 of [Le] with $k = 1$, $a_1 = a$, $b_1 = b$, $h_1 = h$, $p = -1$, and our choices of x_0, g, and ε. Observe that $a_1 b_1^p = aa^{-1} = 1$ and the base case of the recurrence is $\Theta(1)$ as required.

The recurrence above differs from the recurrence in Section 17 only in our choices of a_1 and g. In particular, $s(x) = bx + h(x) \in [1, x)$ for all $x > x_0$.

Observe that $a > 0$ as required by condition 1 of Theorem 2. The function g is non-negative and satisfies Leighton's polynomial-growth condition relative to $\{b\}$ with $c_1 = c_2 = 1$. Satisfaction of the other requirements of condition 1 is inherited from the recurrence in Section 17. Satisfaction of conditions 2, 4(c), and 4(d) is also inherited.

The recurrence of Section 17 has Akra-Bazzi exponent zero and satisfies conditions 4(a) and 4(b). Since $b \cdot \log^{1+\varepsilon} x > 1$ for all $x \geq x_0$, we conclude from $p = -1$ that the recurrence currently under consideration also satisfies conditions 4(a) and 4(b).

Containment of $[bx + h(x), x]$ in $[b, \infty)$ for all $x \geq 1$ is inherited from the recurrence in Section 17. Since g is constant and $[b, \infty) = domain(g)$, condition 3 of Theorem 2 is satisfied with $c_1 = c_2 = 1$. Therefore, the recurrence satisfies the hypothesis of Leighton's Theorem 2.

Our recurrence inherits the following properties from the recurrence in Section 17: $d^*(x)$ is the depth of recursion at x for all $x \in [1, \infty)$, and our recurrence is finitely recursive. Lemma 8.2 implies the recurrence has a unique solution T. Theorem 2 predicts that $T(x) = \Theta(1/x)$.

An easy inductive argument on $d^*(x)$ shows that

$$T(x) = a^{d^*(x)} = b^{d^*(x)}$$

for all $x \in [1, \infty)$. In particular, T is a positive function.

Observe that $\log b < 0$ and $\log(1/b) = |\log b|$ because $0 < b < 1$. Recall from Section 17 that when $x > x_0 + 1$, we have

$$d^*(x) = w(x) = ceiling\left(\log_{1/b}(x/(x_0 + 1))\right) < \frac{\log x}{|\log b|} - 915.43,$$

so

$$T(x) > b^{\frac{\log x}{|\log b|} - 915.43} = b^{-915.43} e^{\frac{(\log b)(\log x)}{|\log b|}} \approx 9900.88 \cdot \mathrm{e}^{-\log x} > \frac{9900}{x},$$

$$d^*(x) \geq \frac{\log x}{|\log b|} - \frac{\log(x_0 + 1)}{|\log b|},$$

and

$$T(x) \leq \left(e^{\frac{(\log b)(\log x)}{|\log b|}}\right)\left(e^{\frac{-(\log b)(\log(x_0+1))}{|\log b|}}\right) = \frac{x_0 + 1}{x} = \frac{10001}{x}.$$

In particular,

$$T(x) = \Theta(1/x)$$

as predicted by Theorem 2. Define

$$L = \inf_{x \in (x_0, x_0+1)} T(x)$$

and

$$M = \inf_{x \in (x_0, x_0+1]} T(x).$$

Recall from Section 15 that $Y \subset (x_0, x_0 + 1]$ and recall from Section 17 that $\sup d^*(Y) = \infty$, so

$$M \leq \inf_{x \in Y} T(x) = \inf_{x \in Y} a^{d^*(x)} = 0.$$

Positivity of T implies $M \geq 0$. Therefore, $M = 0$. Then

$$\min\left(L, T(x_0 + 1)\right) = M = 0$$

because

$$(x_0, x_0 + 1] = (x_0, x_0 + 1) \cup \{x_0 + 1\}.$$

Positivity of T implies

$$T(x_0 + 1) > 0,$$

so $L = 0$.

In the context of our recurrence, the claimed proof of Theorem 2 asserts the existence of a positive real number c_5 that satisfies

$$T(x) \geq \frac{c_5}{x}\left(1 + \frac{1}{\log^{\varepsilon/2} x}\right) > \frac{c_5}{x}$$

for all $x > x_0$, which would imply

$$\frac{c_5}{x_0 + 1}$$

is a positive lower bound for T on $(x_0, x_0 + 1)$ in contradiction to $L = 0$. Therefore, the predicted c_5 does not exist.

The fault lies with the inductive step of Leighton's proof of Theorem 2. The property

$$\inf_{x > x_0} \left(bx + h(x)\right) > e$$

is inherited from the recurrence in Section 17, so the appropriately restricted form of the base case is satisfied as described in Section 16.

19. Problematic and Ill Posed Recurrences

We demonstrated in Sections 13-15 that Theorem 2 of [Le] is false regardless of whether recursion is finite. Later in this section, we show that Lemma 2 of [Le] is also false. Recall that Theorem 2 is applicable to recurrences of the form

$$T(x) = \begin{cases} f(x), & \text{for } x \in [1, x_0] \\ \sum_{i=1}^{k} a_i T\big(b_i x + h_i(x)\big) + g(x), & \text{for } x > x_0 \end{cases}$$

where k is any positive integer and

$$x_0, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k$$

satisfy various conditions. In particular, $x_0 > 1$, the base case $f: [1, x_0] \to \boldsymbol{R}$ is $\Theta(1)$, and g is a non-negative function satisfying Leighton's polynomial-growth condition relative to $\{b_1, \dots, b_k\}$. When $k = 1$, we typically use the shorthand notation a, b, and h for a_1, b_1, and h_1 respectively.

Let

$$R = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g|_I, h_1|_I, \dots, h_k|_I)$$

where $D = [1, \infty)$ and $I = (x_0, \infty)$. The hypothesis of Theorem 2 is apparently intended to imply that R satisfies our definition of a divide-and-conquer recurrence. (This is evident from the claimed proof.) The first eight (of nine) conditions for a semi-divide-and-conquer recurrence are indeed consequences of the hypothesis of Theorem 2. The ninth condition is $b_i x + h_i(x) \in D$, i.e., $b_i x + h_i(x) \geq 1$ (in the context of [Le]), for all $x \in I$ and all $i \in \{1, \dots, k\}$. Unfortunately, satisfaction of that condition is not guaranteed by the hypothesis of Theorem 2, which is also the hypothesis of Lemma 2. A counterexample is provided later in this section.

R is a divide-and-conquer recurrence if it is a semi-divide-and-conquer recurrence that satisfies $b_i x + h_i(x) < x$ for all $x \in I$ and all $i \in \{1, \dots, k\}$, i.e.,

$$b_i x + h_i(x) \in [1, x)$$

for all such x and i. The hypothesis of Theorem 2 and Lemma 2 does not imply that $b_i x + h_i(x) < x$. In particular, the hypothesis is satisfied by some mock divide-and-conquer recurrences such as those recurrences in Section 13 with $x_0 \in [686, 10000)$.

The claimed proof of Theorem 2 fails largely because of the aforementioned issues and the absence of any guarantee that depth of recursion is bounded on each bounded subset of the domain. By Corollary 9.5, the bounded depth condition for a semi-divide-and-conquer recurrence implies the existence of a unique solution, which is locally $\Theta(1)$ when the recurrence is proper and the incremental cost has polynomial growth. The circumstances described in Sections 17 and 18 are thereby avoided. As we shall see, there are additional reasons to require bounded depth of recursion on bounded sets.

The proof of Lemma 2 also fails partly because of the same omission of any guarantee that

$$b_i x + h_i(x) \in [1, x)$$

for all $x > x_0$ and all $i \in \{1, \ldots, k\}$. We will show that Lemma 2 remains false even with the addition of such a guarantee (and an integrability condition). Our obvious replacement in Section 22 for Lemma 2 is applicable to divide-and-conquer recurrences that satisfy the strong ratio condition (and have incremental costs with tame extensions). Theorem 2 relies on the false Lemma 2 but does not assume the existence of

$$0 < \alpha \leq \beta < 1$$

such that

$$\alpha x \leq b_i x + h_i(x) \leq \beta x$$

for all $x > x_0$ and all $i \in \{1, \ldots, k\}$.

Our most fundamental replacement in Sections 20 for Leighton's Theorem 2 is a statement about locally $\Theta(1)$ solutions of mildly constrained semi-divide-and-conquer recurrences. The recurrences need not satisfy either the bounded depth or ratio conditions. Furthermore, they need not be proper, i.e., our replacement is applicable to some mock divide-and-conquer recurrence. However, each locally $\Theta(1)$ solution is also the solution of an auxiliary divide-and-conquer recurrence that satisfies the bounded depth and strong ratio conditions and some conditions analogous to the hypothesis of Leighton's Theorem 2. The auxiliary recurrence is obtained by a suitable extension of the base case. See Section 20 for more information.

Failure of inductive step and partition of the domain. The claimed proof of Leighton's Theorem 2 uses a sequence $I_0, I_1, I_2, \ldots$ of disjoint, bounded, non-empty intervals with $I_0 = [1, x_0]$ and

$$[1, \infty) = \bigcup_{j=0}^{\infty} I_j \, ,$$

so $\{I_j : j \in \boldsymbol{N}\}$ is a partition of $[1, \infty)$. Furthermore, sup $I_j = \inf I_{j+1}$ for all $j \in \boldsymbol{N}$. The definition of the partition is contained in the proof of Theorem 1 of [Le]. In this section, we ignore the precise definition of the intervals and consider only their properties described above. Leighton's argument uses induction on the index of the interval containing x. The inductive step implicitly assumes $b_i x + h_i(x)$ maps I_n into

$$\bigcup_{j=0}^{n-1} I_j$$

for all $n \in \boldsymbol{Z}^+$and all $i \in \{1, \ldots, k\}$. Assuming the hypothesis of Theorem 2, the existence of such a partition requires R to be a divide-and-conquer recurrence with depth of recursion at most n on the interval I_n. If S is any bounded subset of $[1, \infty)$, then sup S is contained in I_d for some non-negative integer d, so

$$S \subseteq \bigcup_{j=0}^{d} I_j$$

and the depth of recursion is at most d on S. Therefore, the bounded depth condition must be satisfied for the inductive step to possibly be valid.

In Section 13, we described a family of semi-divide-and-conquer recurrences (parameterized by a choice of $x_0 \in [686,10000]$) that satisfy the hypothesis of Leighton's Theorem 2 although each member of the family has uncountably many solutions that do not satisfy the proposition's conclusion. Those recurrences satisfy

$$bx + h(x) > 10000$$

for all $x > 10000$. When $x_0 = 10000$, there is no possible choice of I_1 for which $bx + h(x)$ maps I_1 into I_0. When $x_0 \in [686, 10000)$, Lemma 13.1 implies

$$bx + h(x) \geq x$$

for all $x \in (x_0, 10000]$ (with equality if and only if $x = 10000$); in particular, the recurrence is a mock divide-and-conquer recurrence. The inductive step fails for the entire family of recurrences in Section 13. All of them are infinitely recursive, so none of them satisfy the bounded depth condition.

The finitely recursive counterexample in Section 15 to Leighton's Theorem 2 is a divide-and-conquer recurrence that violates the bounded depth condition. For example, depth of recursion is unbounded on $(x_0, x_0 + 1)$. There is no partition of the domain with the claimed properties, and the inductive step fails.

The divide-and-conquer recurrences described in Sections 17 and 18 also violate the bounded depth condition and lack partitions of their domains with the claimed properties.

Later in this section, we shall exhibit an ill posed recurrence that satisfies the hypothesis of Theorem 2 but has the property that $bx + h(x) < 1$ for some $x > x_0$. There is no partition with the assumed properties, and the inductive step of the proof fails again.

Violation of the strong ratio condition and failure of Lemma 2. Leighton's Lemma 2 asserts the existence of positive real numbers c_3 and c_4 with the property that

$$c_3 g(x) \leq x^p \int_{b_i x + h_i(x)}^{x} \frac{g(u)}{u^{p+1}} du \leq c_4 g(x)$$

for all $x \geq 1$ and all $i \in \{1, \dots, k\}$. Although the Lemma does not define p, Leighton obviously intends for p to be the same real number that appears in the statement of Theorem 2, so that

$$\sum_{i=1}^{k} a_i b_i^p = 1.$$

The claimed proof of Theorem 2 uses Lemma 2 with $x > x_0$ but does not use Lemma 2 for $x \in [1, x_0]$. Therefore, we shall confine our attention to $x > x_0$.

Now consider any recurrence with $x_0 \in [686, 10000)$ in the family of counterexamples in Section 13 to Theorem 2. The function g for the recurrence is positive. As before, Lemma 13.1 implies

$$bx + h(x) \geq x$$

for all $x \in (x_0, 10000]$, i.e., the lower limit of integration is greater than or equal to the upper limit of integration. The integrand in Lemma 2 is positive for all such x, so the oriented integral is non-positive, which implies $c_3 g(x) \leq 0$ in contradiction to the positivity of c_3 and g. (When $x = 10000$, the upper and lower limits are the same and the integral is zero. The upper and lower limits are distinct when $x \neq 10000$.) The recurrence is a counterexample to Lemma 2, which is false.

The recurrence in Section 13 with $x_0 = 10000$ and the finitely recursive counterexample in Section 15 to Theorem 2 satisfy the conditions of Theorem 2 and are divide-and-conquer recurrences with

$$bx + h(x) \in [1, x)$$

for all $x > x_0$. They share the properties that $g(x) = 1$ for all $x \in domain(g)$.

We claim that Lemma 2 is false for both recurrences. Lemmas 13.1(3), 13.22, and 15.1 imply that each recurrence satisfies

$$\inf_{x_0 < bx + h(x) < x < x_0 + 1} \left(x - \left(bx + h(x) \right) \right) = 0.$$

Pick either recurrence and define

$$U = \frac{\max({x_0}^p, (x_0+1)^p)}{\min({x_0}^{p+1}, (x_0+1)^{p+1})},$$

so $U > 0$. There exists $z \in (x_0, x_0+1)$ such that

$$x_0 < bz + h(z) < z$$

and

$$z - \big(bz + h(z)\big) < \frac{c_3}{U}.$$

According to Lemma 2:

$$c_3 = c_3 g(z) \le z^p \int_{bz+h(z)}^{z} \frac{1}{u^{p+1}}\,du < U \cdot \Big(z - \big(bz + h(z)\big)\Big) < c_3.$$

We obtain the contradiction $c_3 < c_3$, which demonstrates that Lemma 2 is false for both of the recurrences.

The claimed proof of Lemma 2 consists of the following statement: "The proof is identical to that for Lemma 1 except that we use constraint 3 above in place of the polynomial-growth condition of Section 2."

An obvious translation of the claimed proof of Lemma 2 into a proof of a true version of the lemma for $x > x_0$ includes the requirement that

$$\min_{1\le i\le k}\left(\inf_{x>x_0}\left(b_i + \frac{h_i(x)}{x}\right)\right) > 0 \quad\text{and}\quad \max_{1\le i\le k}\left(\sup_{x>x_0}\left(b_i + \frac{h_i(x)}{x}\right)\right) < 1,$$

i.e.

$$\inf_{x>x_0}\left(b_i + \frac{h_i(x)}{x}\right) > 0 \quad\text{and}\quad \sup_{x>x_0}\left(b_i + \frac{h_i(x)}{x}\right) < 1$$

for all $i \in \{1, \dots, k\}$. For a semi-divide-and-conquer recurrence with recursion interval (x_0, ∞), the upper bound above is equivalent to the ratio condition (which implies the recurrence is proper) and the combination of inequalities above is equivalent to the strong ratio condition. As previously explained, the counterexamples in Section 13 and 15 are semi-divide-and-conquer recurrences that violate the bounded depth condition. (The example in Section 15 is proper as is the recurrence in Section 13 when $x_0 = 1000$). Lemma 9.6 implies they also violate the ratio condition.

Depending on interpretation, the existence of a positive lower bound might be considered a consequence of the hypothesis of Leighton's Theorem 2 except perhaps when $p = 0$. Theorem 2 assumes $0 < b_i < 1$ and $x_0 \ge 1/b_i$ for all $i \in \{1, \dots, k\}$, so $x_0 > 1$ and $\log x_0 > 0$. Define positive real-valued functions $\lambda_1, \dots, \lambda_k : [x_0, \infty) \to \boldsymbol{R}^+$ by

$$\lambda_i(x) = b_i \log^{1+\varepsilon} x$$

where $\varepsilon > 0$ satisfies condition 4(a) of Theorem 2. Each λ_i is continuous and increasing. Furthermore,

$$\lim_{x\to\infty} \lambda_i(x) = \infty.$$

Suppose $\lambda_j(x_0) \le 1$ for some $j \in \{1, \dots, k\}$. Then there exists $t \ge x_0$ with $\lambda_j(t) = 1$. Observe that

$$t > b_j t \ge b_j x_0 \ge 1.$$

In particular, $\log t > 0$ and

$$\log\left(b_j t + \frac{t}{\log^{1+\varepsilon} t}\right) > \log(b_j t) \ge 0.$$

Then condition 4(a) implies

$$0^p = \left(1 - \frac{1}{b_j \log^{1+\varepsilon} t}\right)^p \ge \left(1 + \frac{1}{\log^{\varepsilon/2}\left(b_j t + \frac{t}{\log^{1+\varepsilon} t}\right)}\right)^{-1} \left(1 + \frac{1}{\log^{\varepsilon/2} t}\right) > 0.$$

In particular, 0^p must be defined and positive, which is false unless $p = 0$ and we adopt Knuth's convention that $0^0 = 1$. If $p \ne 0$ or we consider 0^0 to be undefined, then

$$b_i \log^{1+\varepsilon} x_0 > 1$$

for all $i \in \{1, \dots, k\}$. If ε also satisfies condition 2 of Theorem 2, then

$$\inf_{x > x_0} \left(b_i + \frac{h_i(x)}{x}\right) \ge \inf_{x > x_0} \left(b_i - \frac{1}{\log^{1+\varepsilon} x}\right) = b_i \left(1 - \frac{1}{b_i \log^{1+\varepsilon} x_0}\right) > 0$$

for each index i.

With some small modifications (the *technical condition* in Section 20) to the inequalities in the hypothesis of Theorem 2, satisfaction of the strong ratio condition is guaranteed. See the proof of Lemma 20.9.

Example of range of dependency not contained in domain of recurrence. Consider the recurrence

$$T(x) = aT\big(bx + h(x)\big) + 1$$

for

$$x > x_0 = e^{10} + 1 \approx 22027,$$

and

$$T(x) = 1$$

for $1 \le x \le x_0$ where $b = 1/100$, $a = b^2$, and the function $h: [1, \infty) \to \boldsymbol{R}$ is defined by

$$h(x) = -\frac{x}{\log^2 x}$$

for $x > x_0$, and $h(x) = 0$ for $1 \le x \le x_0$. We claim the hypothesis of Theorem 2 is satisfied by the recurrence with $p = -2$, $\varepsilon = 1$, and $g: \boldsymbol{R} \to \boldsymbol{R}$ defined by $g(x) = 1$. Observe that $ab^p = b^2 b^{-2} = 1$ as required. Satisfaction of conditions 1, 2, and 3 of Theorem 2 is immediately obvious. Observe that

$$\log^{\varepsilon/2} x = \sqrt{\log x} \ge \sqrt{\log x_0} > \sqrt{10} > 3$$

for all $x \ge x_0$, so

$$\frac{1}{2}\left(1 + \frac{1}{\log^{\varepsilon/2} x}\right) < 1$$

and

$$2\left(1 - \frac{1}{\log^{\varepsilon/2} x}\right) > 1,$$

so conditions 4(c) and 4(d) of Theorem 2 are satisfied. For all such x, we have

$$\log^2 x \ge \log^2 x_0 > 100,$$

$$b \log^{1+\varepsilon} x = b \log^2 x > 100b = 1,$$

$$\left(1 - \frac{1}{b \log^{1+\varepsilon} x}\right)^p = \left(1 - \frac{1}{b \log^2 x}\right)^{-2} > 1,$$

$$0 < \left(1 + \frac{1}{b \log^{1+\varepsilon} x}\right)^p = \left(1 + \frac{1}{b \log^2 x}\right)^{-2} < 1,$$

and

$$e < bx_0 \le bx + \frac{x}{\log^{1+\varepsilon} x} = \frac{x}{100} + \frac{x}{\log^2 x} < \frac{x}{50} < x,$$

so

$$1 < \log\left(bx + \frac{x}{\log^{1+\varepsilon} x}\right) < \log x.$$

Then

$$\left(1 - \frac{1}{b \log^{1+\varepsilon} x}\right)^p \left(1 + \frac{1}{\log^{\varepsilon/2}\left(bx + \frac{x}{\log^{1+\varepsilon} x}\right)}\right) > 1 + \frac{1}{\log^{\varepsilon/2} x}$$

and

$$\left(1 + \frac{1}{b \log^{1+\varepsilon} x}\right)^p \left(1 - \frac{1}{\log^{\varepsilon/2}\left(bx + \frac{x}{\log^{1+\varepsilon} x}\right)}\right) < 1 - \frac{1}{\log^{\varepsilon/2} x},$$

so conditions 4(a) and 4(b) are satisfied. Therefore, the hypothesis of Theorem 2 is satisfied. Define $r\colon (x_0, \infty) \to \boldsymbol{R}$ by

$$r(x) = bx + h(x),$$

so r is differentiable with derivative

$$r'(x) = \frac{(\log x)(\log^2 x - 100) + 200}{100 \log^3 x}.$$

Observe that

$$\log x > 10$$

for all $x > x_0$, so

$$r'(x) > 0$$

for each such x, i.e., r is increasing on (x_0, ∞). Let

$$m = \lim_{x \to x_0^+} \left(r(x)\right),$$

so

$$m = bx_0 - \frac{x_0}{\log^2 x_0} = \frac{e^{10} + 1}{100} - \frac{e^{10} + 1}{\log^2(e^{10} + 1)} \approx 0.002.$$

Furthermore,

$$\lim_{x \to \infty} \left(r(x)\right) = \left(\lim_{x \to \infty} \left(b - \frac{1}{\log^2 x}\right)\right) \left(\lim_{x \to \infty} (x)\right) = b \cdot \infty = \infty.$$

Therefore, the range of the continuous, increasing function r is the interval (m, ∞), which is not contained in the domain, $[1, \infty)$, of T. We conclude that the recurrence has no solution. The recurrence fulfills promises made earlier in this section and in Sections 0, 7, and 16 including

$$\inf_{x > x_0} (r(x)) < 1 \quad \text{and} \quad r(y) = 1$$

for some $y > x_0$. Let $t \in (m, 1)$, so $1, t \in range(r)$. Observe that $t \in (0,1)$, so $\log t$ is a negative real number. Leighton's inductive argument (see Section 16) requires that

$$\frac{1}{\log^{\varepsilon/2} 1} \quad \text{and} \quad \frac{1}{\log^{\varepsilon/2} t}$$

are real. Of course, neither expression represents a real number.

Nonexistence of a solution is illustrated by the C# code on the next page. The method `Test` of the `Example` class causes an `ArgumentOutOfRangeException` to be thrown by the method `T`. (We ignore the issue of floating-point rounding.)

```
public static class Example
{
    const double a = 0.0001;
    const double b = 0.01;
    static readonly double x0 = Math.Exp(10.0) + 1;

    public static double Test()
    {
        return T(x0 + 1);
    }

    static double T(double x)
    {
        if (x > x0)
        {
            return a * T(b * x + h(x)) + 1;
        }
        else if (x >= 1)
        {
            return 1;
        }
        else
        {
            throw new ArgumentOutOfRangeException();
        }
    }

    static double h(double x)
    {
        if (x > x0)
        {
            double logX = Math.Log(x);
            return -x / (logX * logX);
        }
        else if (x >= 1)
        {
            return 0;
        }
        else
        {
            throw new ArgumentOutOfRangeException();
        }
    }
}
```

20. Replacements for Leighton's Theorem 2

Our main results are Theorems 20.11 and 21.2 along with Corollaries 20.12 and 20.13. Together they form a convenient replacement for the false Theorem 2 of [Le]. They are applicable to certain recurrences with sufficiently linear dependencies:

Definition. A semi-divide-and-conquer recurrence has *low noise* if either the recursion set is bounded, or for each noise term h there exists $c > 1$ such that

$$|h(x)| = O\left(\frac{x}{\log^c x}\right).$$

The Big-O relationship in the definition of low noise requires the recursion set I to be unbounded above. By definition of a semi-divide-and-conquer recurrence, I has a positive lower bound. Thus I is bounded if and only if I has a finite upper bound, i.e., I is unbounded if and only if I is unbounded above. Of course, the expression $x/\log^c x$ represents a positive real number for all $x \in I \cap (1, \infty)$.

The definition above refers to $|h(x)|$ instead of $h(x)$ because our definition of Big-O notation requires the related functions to be asymptotically non-negative. Our interpretation of the asymptotic relationship for $|h(x)|$ is compatible with other sources (such as [Kn]) that have no such requirement.

Of course, there exists a uniform choice for the exponent c in the definition of low noise when the recursion set is unbounded: If a semi-divide-and-conquer recurrence

$$(D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k)$$

with an unbounded recursion set has low noise, then there exist $c_1, \dots, c_k > 1$ such that

$$|h_i(x)| = O\left(\frac{x}{\log^{c_i} x}\right)$$

for all $i \in \{1, \dots, k\}$. Then

$$|h_i(x)| = O\left(\frac{x}{\log^{c} x}\right)$$

for all such i where $c = \min(c_1, \dots, c_k)$. Observe that $c > 1$ and

$$\lim_{x\to\infty} \frac{h_i(x)}{x} = 0$$

for all $i \in \{1, \dots, k\}$. As explained in Section 7, each dependency of such a recurrence has a unique representation of the form $bx + h(x)$ that is consistent with low noise.

The following definition and lemma provide a characterization of low noise that is more in the spirit of [Le]. Section 30 discusses an example in [Le] that demonstrates the motivation for $c > 1$ in the definition of low noise and $\varepsilon > 0$ in the definition below.

Definition. A semi-divide-and-conquer recurrence with recursion set I satisfies *Leighton's noise condition* on J relative to a positive number ε if J is a subset of $I \cap (1, \infty)$ and

$$|h(x)| \le \frac{x}{\log^{1+\varepsilon} x}$$

for each noise term h and all $x \in J$.

The requirement above that $J \subseteq I \cap (1, \infty)$ guarantees that $\log x$ and $x/(\log^{1+\varepsilon} x)$ are defined as positive real numbers for all $x \in J$.

Lemma 20.1. A semi-divide-and-conquer recurrence R with unbounded recursion set I has low noise if and only if there exists $\varepsilon > 0$ and a non-empty upper subset J of I such that R satisfies Leighton's noise condition on J relative to ε.

Proof. If R satisfies Leighton's noise condition on some non-empty upper subset J of I relative to some $\varepsilon > 0$, then

$$|h(x)| \le \frac{x}{\log^{1+\varepsilon} x}$$

for each noise term h and all $x \in J$. We conclude from $\sup I = \infty$ that

$$|h(x)| = O\left(\frac{x}{\log^{1+\varepsilon} x}\right)$$

for each such h, i.e., R has low noise.

We now prove the converse. Suppose R has low noise with noise terms $h_1, \dots, h_k$. Since I is unbounded, there exists $c > 1$ along with $M_1, \dots, M_k \in \boldsymbol{R}^+$ and non-empty upper subsets $H_1, \dots, H_k$ of $I \cap (1, \infty)$ such that

$$|h_i(x)| \le M_i \frac{x}{\log^c x}$$

for all $x \in H_i$ and all $i \in \{1, \dots, k\}$.

We claim $H_i \cap H_j \in \{H_i, H_j\}$ for all $i, j \in \{1, \dots, k\}$: If $H_i - H_j \neq \emptyset$ and $H_j - H_i \neq \emptyset$, there exist $y \in H_i - H_j$ and $z \in H_j - H_i$, so $y > z > y$, which is a contradiction. Either $H_i - H_j = \emptyset$ and $H_i \cap H_j = H_i$, or $H_j - H_i = \emptyset$ and $H_i \cap H_j = H_j$. The claim follows.

Define

$$H = \bigcap_{i=1}^{k} H_i.$$

An obvious inductive argument implies $H \in \{H_1, \dots, H_k\}$. In particular, H is a non-empty upper subset of $I \cap (1, \infty)$, which is an upper subset of I. Therefore, H is an upper subset of I and $\sup H = \sup I = \infty$. Observe that

$$|h_i(x)| \leq U \frac{x}{\log^c x}$$

for all $x \in H$ and all $i \in \{1, \dots, k\}$ where $U = \max\{M_1, \dots, M_k\}$. Define $\varepsilon = (c-1)/2$, so $\varepsilon > 0$ and

$$|h_i(x)| \leq \frac{U}{\log^\varepsilon x} \cdot \frac{x}{\log^{1+\varepsilon} x}$$

for all $x \in H$ and each index i. Since $\sup H = \infty$ and

$$\lim_{x \to \infty} \frac{U}{\log^\varepsilon x} = 0,$$

there exists a non-empty upper subset J of H such that

$$|h_i(x)| < \frac{x}{\log^{1+\varepsilon} x}$$

for all $x \in J$ and all $i \in \{1, \dots, k\}$. Of course, $J \subseteq H \subseteq I \cap (1, \infty)$. Therefore, R satisfies *Leighton's noise condition* on J relative to ε. Furthermore, J is an upper subset of I because J is an upper subset of H, which is an upper subset of I. □

We are primarily interested in *admissible* recurrences:

Definition. An *admissible recurrence* is a semi-divide-and-conquer recurrence with low noise whose incremental cost has a tame extension.

By definition, the incremental cost's domain (i.e., the recursion set) is positive and non-empty; the domain of a tame function is a non-empty, positive interval. The incremental cost of an admissible recurrence has polynomial growth by Lemma 2.2(2). Similarly, Lemma 10.1(2) implies the incremental cost of a semi-divide-and-conquer recurrence has a tame extension if and only if there is such an extension to the minimum interval containing the recursion set. The incremental cost of a semi-divide-and-conquer recurrence with an interval for its recursion set has a tame extension if and only if the incremental cost is tame.

The requirements for an admissible recurrence are analogous to condition (2) and most of condition (1) of Theorem 2 in [Le]. Later in this section, we define the *technical condition,* which is analogous to condition (4) and part of condition (1). We have no use for condition (3). Our closest analog of the hypothesis of Theorem 2 is the *modified Leighton hypothesis*, which is also defined later in this section.

Definition. Let I be the recursion set of a semi-divide-and-conquer recurrence and suppose g is a tame extension of the incremental cost (so $domain(g)$ is an interval containing I). The *Akra-Bazzi estimate* for the recurrence relative to g is the function $A: I \to \boldsymbol{R}^+$ defined by

$$A(x) = x^p \left(1 + \int_{x_0}^{x} \frac{g(u)}{u^{p+1}} du\right)$$

where $x_0 = \inf I$ and p is the Akra-Bazzi exponent (defined at the end of Section 11). If $x_0 \notin domain(g)$ (which implies $x_0 = \inf domain(g)$), the integral above is interpreted as the improper integral

$$\lim_{t \to x_0^+} \int_{t}^{x} \frac{g(u)}{u^{p+1}} du.$$

Let g, x_0, and p be as in the definition above. Corollary 10.3 implies the function $u \mapsto g(u)/u^{p+1}$ on $domain(g)$ is tame (in particular, it is locally Riemann integrable). By definition of a semi-divide-and-conquer recurrence, x_0 is positive. We conclude from Lemma 10.5 that the Akra-Bazzi integral converges when it is improper. Lemma 10.1(1) (or Lemma 2.2(1)) implies the integrand is non-negative, so the Akra-Bazzi estimate is indeed a positive function.

Definition. Suppose g is a tame extension of the incremental cost of a semi-divide-and-conquer recurrence with recursion set I. Let A be the Akra-Bazzi estimate for the recurrence relative to g. A solution T of the recurrence satisfies the *strong Akra-Bazzi condition* (relative to the recurrence and g) if there exist positive real numbers λ_1 and λ_2 such that

$$\lambda_1 A(x) \le T(x) \le \lambda_2 A(x)$$

for all x in I. A solution T satisfies the w*eak Akra-Bazzi condition* (relative to the recurrence and g) if I is unbounded and

$$T(x) = \Theta\big(A(x)\big).$$

The weak Akra-Bazzi condition is similar to the conclusion of Theorem 2 of [Le] (the lower limit of integration differs). We are more interested in the strong Akra-Bazzi condition, which is a weaker version of the inductive hypothesis in the claimed proof of Theorem 2 in [Le]. Of course, the strong Akra-Bazzi condition implies the weak Akra-Bazzi condition when the recursion set is unbounded.

Flexibility with lower limit of integration. Let R be a semi-divide-and-conquer recurrence with recursion set I and Akra-Bazzi exponent p. Further suppose T is a solution of R and g is a tame extension of the incremental cost of R. Let

$$c \in [\inf domain(g), \inf I] - \{0\},$$

so $(c, x] \subseteq domain(g)$ for all $x \in I$. Define $B: I \to \boldsymbol{R}^+$ by

$$B(x) = x^p \left(1 + \int_c^x \frac{g(u)}{u^{p+1}} du\right).$$

As before: The integrand above (defined on $domain(g)$) is tame by Corollary 10.3 (in particular, it is locally Riemann integrable). It is non-negative by Lemma 10.1(1) (or Lemma 2.2(1)), so B is a positive function as claimed. If $c \notin domain(g)$, the integral above is improper and Lemma 10.5 guarantees convergence.

Lemma 10.8 implies T satisfies the strong Akra-Bazzi condition relative to R and g if and only if there exist positive real numbers α and β such that

$$\alpha B(x) \leq T(x) \leq \beta B(x)$$

for all $x \in I$. Similarly, T satisfies the weak Akra-Bazzi condition relative to R and g if and only if $T(x) = \Theta\big(B(x)\big)$.

We make some trivial observations before proceeding to deeper water:

Lemma 20.2. Let R be a semi-divide-and-conquer recurrence whose incremental cost has a tame extension g. The Akra-Bazzi estimate for R relative to g is locally $\Theta(1)$.

Proof. Let A be the Akra-Bazzi estimate for R relative to g, and let $x_0 = \inf I$ where I is the recursion set of R. By definition of a semi-divide-and-conquer recurrence, x_0 is positive. Let $J = domain(g)$. Positivity of x_0 and containment of I in the interval J imply $x_0 \in J \cup (\{\inf J\} - \{0\})$. Lemma 10.6 implies the function

$$x \mapsto x^p \left(1 + \int_{x_0}^x \frac{g(u)}{u^{p+1}} du\right)$$

on $J \cap [x_0, \infty)$ is locally $\Theta(1)$, so its restriction, A, to I is also locally $\Theta(1)$. □

Lemma 20.3. Suppose T is a solution of a semi-divide-and-conquer recurrence R whose incremental cost has a tame extension g. Let A be the Akra-Bazzi estimate for R relative to g, and let J be a subset of the recursion set of R. If there exist $\lambda_1, \lambda_2 \in \boldsymbol{R}^+$ such that

$$\lambda_1 A(x) \leq T(x) \leq \lambda_2 A(x)$$

for all $x \in J$, then the restriction of T to J is locally $\Theta(1)$.

Proof. Let W be a bounded subset of J, so W is contained in the recursion set. Lemma 20.2 implies A is locally $\Theta(1)$. There exist $c_1, c_2 \in \boldsymbol{R}^+$ such that

$$c_1 \leq A(x) \leq c_2$$

for all $x \in W$. Then

$$\lambda_1 c_1 \leq T(x) \leq \lambda_2 c_2$$

for all such x, so $T|_W = \Theta(1)$. Therefore, $T|_J$ is locally $\Theta(1)$. □

Corollary 20.4. Let R be a semi-divide-and-conquer recurrence whose incremental cost has a tame extension g. If a solution T of R satisfies the strong Akra-Bazzi condition relative to R and g, then T is locally $\Theta(1)$.

Proof. Lemma 20.3 implies the restriction of T to the recursion set of R is locally $\Theta(1)$. Lemma 9.1 implies T is locally $\Theta(1)$. □

Corollary 20.5. Let R be a semi-divide-and-conquer recurrence whose incremental cost has a tame extension g. If a solution T of R satisfies the weak Akra-Bazzi condition relative to R and g, then the restriction of T to H is locally $\Theta(1)$ for some non-empty upper subset H of the recursion set of R.

Proof. Let I be the recursion set of R. By definition of the weak Akra-Bazzi condition, I is unbounded and

$$T(x) = \Theta\big(A(x)\big)$$

where A is the Akra-Bazzi estimate for R relative to g. There exist $\lambda_1, \lambda_2 \in \boldsymbol{R}^+$ and a non-empty upper subset H of I such that

$$\lambda_1 A(x) \leq T(x) \leq \lambda_2 A(x)$$

for all $x \in H$. Lemma 20.3 implies the restriction of T to H is locally $\Theta(1)$. □

Weak and strong Akra-Bazzi conditions are not equivalent. Although the strong Akra-Bazzi condition implies the weak Akra-Bazzi condition when the recursion set is unbounded, the converse is false. Sections 17 and 18 contain examples of proper, finitely recursive, admissible recurrences (with unbounded recursion sets) whose unique solutions satisfy the weak Akra-Bazzi condition but are not $\Theta(1)$ on $(10000,10001)$.

The solutions are therefore not locally $\Theta(1)$. They violate the strong Akra-Bazzi condition by Corollary 20.4.

Some admissible recurrences violate both Akra-Bazzi conditions. Section 15 describes a proper, finitely recursive, admissible recurrence with an unbounded recursion set whose unique solution violates both Akra-Bazzi conditions.

Dangerous bend. In the presence of infinite recursion, an admissible recurrence may have a solution that satisfies the strong Akra-Bazzi condition or the weak Akra-Bazzi condition, while having other solutions that satisfy neither. Section 13 defines a family of infinitely recursive admissible recurrences parameterized by a choice of x_0 in the interval $[686,10000]$ and a $\Theta(1)$ base case $f\colon [1, x_0] \to \boldsymbol{R}$. (The only proper recurrences are the ones with $x_0 = 10000$.) The recursion set is (x_0, ∞). Now consider any recurrence in the family. The constant function $g(x) = 1$ on $(0, \infty)$ is a tame extension of the incremental cost. (Here the domain of g is chosen for consistency of notation with Section 13.) The Akra-Bazzi estimate for the recurrence relative to g is the function $A\colon (x_0, \infty) \to \boldsymbol{R}$ defined by

$$A(x) = \frac{1}{x}\left(1 + \int_{x_0}^{x} du\right) = 1 + \frac{1 - x_0}{x},$$

so

$$A(x) < 1.$$

The function A has derivative

$$A'(x) = \frac{x_0 - 1}{x^2} > 0,$$

so A (and its obvious continuous extension to $[x_0, \infty)$) is an increasing function, which implies

$$A(x) > 1 + \frac{1 - x_0}{x_0} = \frac{1}{x_0}.$$

The constant function $x \mapsto 100$ on (x_0, ∞) can be extended to a solution T of the recurrence and satisfies

$$100 \cdot A(x) < T(x) < 100 x_0 \cdot A(x)$$

for all $x > x_0$. Therefore, T satisfies the strong Akra-Bazzi condition relative to the recurrence and g. As shown in Section 13, T is not the unique solution of the recurrence. Infinitely many other solutions are unbounded on every non-empty open subset of the recursion interval and therefore violate both Akra-Bazzi conditions by Corollaries 20.4 and 20.5.

Some recurrences in the family have $range(f) = \{100\}$. For such a recurrence, the solution T described above is constant.

Criticality of base case's domain. A much different family of admissible recurrences is obtained from the family in Section 13 by specifying that $x_0 > 10000$. The resulting recurrences satisfy the ratio condition (in particular, they are proper). Lemma 9.6 implies each such recurrence satisfies the bounded depth condition (and is therefore finitely recursive) and has a unique solution. In particular, each recurrence in the family with the constant base case $x \mapsto 100$ on $[1, x_0]$ has the constant function $x \mapsto 100$ on $[1, \infty)$ as its unique solution. Theorem 2 of [Le] contains the phrase "x_0 is chosen to be a large enough constant". As we have seen, a recurrence can be dramatically altered by modification of the base case.

Lemma 20.6. If T is a locally $\Theta(1)$ solution of an admissible recurrence

$$R = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k),$$

and J is a non-empty upper subset of I, then

$$S = \left(D, J, a_1, \dots, a_k, b_1, \dots, b_k, T|_{D-J}, g|_J, h_1|_J, \dots, h_k|_J\right)$$

is also an admissible recurrence with T as a solution. Furthermore, if g^* is a tame extension of g, then T satisfies the strong Akra-Bazzi condition relative to R and g^* if and only if T satisfies the strong Akra-Bazzi condition relative to S and g^*.

Proof. Lemma 9.7 implies S is a semi-divide-and-conquer recurrence with T as a solution. By definition of an admissible recurrence, g has a tame extension, which is also an extension of $g|_J$. If I is bounded, then J is bounded, which implies S has low noise and is therefore admissible. If I is unbounded, then J is unbounded and there exists $c > 1$ such that

$$|h_i(x)| = O\left(\frac{x}{\log^c x}\right)$$

for all $i \in \{1, \dots, k\}$. Since J is a non-empty upper subset of I,

$$\left|h_i|_J(x)\right| = O\left(\frac{x}{\log^c x}\right)$$

for each index i, which implies S has low noise and is therefore admissible.

Let g^* be a tame extension of g. Of course, g^* is also an extension of $g|_J$. Let A and B be the Akra-Bazzi estimates for R and S respectively (relative to g^*). Define $A^*: D \to \boldsymbol{R}$ and $B^*: D \to \boldsymbol{R}$ by

$$A^* = \begin{cases} A(x), & \text{for } x \in I \\ T(x), & \text{for } x \in D - I \end{cases}$$

and

$$B^* = \begin{cases} B(x), & \text{for } x \in J \\ T(x), & \text{for } x \in D - J. \end{cases}$$

By definition of a semi-divide-and-conquer recurrence, the function f and $T|_{D-J}$ are $\Theta(1)$. Furthermore, $\inf I$ and $\inf J$ are positive. Observe that R and S have the same

Akra-Bazzi exponents. Lemma 10.9 implies there exists positive real numbers λ_1 and λ_2 such that

$$\lambda_1 A^*(x) \le B^*(x) \le \lambda_2 A^*(x)$$

for all $x \in D$.

Suppose T satisfies the strong Akra-Bazzi condition relative to R and g^*. We will show that T satisfies the strong Akra-Bazzi condition relative to S and g^*. There exist positive numbers α_1 and α_2 such that

$$\alpha_1 A(x) \le T(x) \le \alpha_2 A(x)$$

for all $x \in I$. Since $J \subseteq I$, we have $A(x) = A^*(x)$ for all $x \in J$. For such x, we have

$$\frac{\alpha_1}{\lambda_2} B(x) = \frac{\alpha_1}{\lambda_2} B^*(x) \le \alpha_1 A^*(x) = \alpha_1 A(x) \le T(x)$$

and

$$T(x) \le \alpha_2 A(x) = \alpha_2 A^*(x) \le \frac{\alpha_2}{\lambda_1} B^*(x) = \frac{\alpha_2}{\lambda_1} B(x).$$

The quantities α_1/λ_2 and α_2/λ_1 are positive real numbers, so T satisfies the strong Akra-Bazzi condition relative to S and g^* as claimed.

The converse: Suppose T satisfies the strong Akra-Bazzi condition relative to S and g^*. We will show that T satisfies the strong Akra-Bazzi condition relative to R and g^*. There exist positive real numbers β_1 and β_2 such that

$$\beta_1 B(x) \le T(x) \le \beta_2 B(x)$$

for all $x \in J$. Let $L = \min\{\beta_1, 1\}$ and $U = \max\{\beta_2, 1\}$. Then

$$LB^*(x) = LB(x) \le \beta_1 B(x) \le T(x) \le \beta_2 B(x) \le UB(x) = UB^*(x)$$

for all $x \in J$, and

$$LB^*(x) \le B^*(x) = T(x) = B^*(x) \le UB^*(x)$$

for all $x \in D - J$. Therefore,

$$LB^*(x) \le T(x) \le UB^*(x)$$

for all $x \in D$. For all $x \in I$, we have

$$\lambda_1 LA(x) = \lambda_1 LA^*(x) \le LB^*(x) \le T(x)$$

and

$$T(x) \le UB^*(x) \le \lambda_2 UA^*(x) = \lambda_2 UA(x).$$

The quantities $\lambda_1 L$ and $\lambda_2 U$ are positive real numbers, so T satisfies the strong Akra-Bazzi condition relative to R and g^* as claimed. □

The bounded depth and strong ratio conditions and locally $\Theta(1)$ solutions of admissible recurrences with unbounded recursion sets. If the admissible recurrence R of Lemma 20.6 has an unbounded recursion set, then Lemma 9.8 implies there exists a choice of J such that S satisfies the strong ratio condition. Then Lemma 9.6 implies S satisfies the bounded depth condition and has T as its unique solution.

An infinitely recursive, proper, admissible recurrence R with a bounded recursion set and a positive constant solution T (and infinitely many solutions that are not $\Theta(1)$) such that the recurrence S of Lemma 20.6 does not satisfy the bounded depth condition for any choice of J. Let I be the open interval $(1,2)$ and define a bijection $r: I \to I$ by

$$r(x) = (x-1)^2 + 1,$$

so $r(x) < x$ for all $x \in I$. Let

$$R = (D, I, a, b, f, g, h)$$

where $D = [1,2)$, $a = 1/2$, $b \in (0,1)$, $f: \{1\} \to \{1\}$, $g: I \to \{1/2\}$, and $h: I \to \boldsymbol{R}$ is defined by $h(x) = r(x) - bx$. Observe that

$$bx + h(x) = r(x) \in (1, x) = I \cap (-\infty, x) \subset D \cap (-\infty, x)$$

for all $x \in I$, so R is a (proper) divide-and-conquer recurrence that is infinitely recursive at each such x. The recurrence R has low noise because the recursion set, I, is bounded. The incremental cost, g, is its own tame extension. Therefore, R is an admissible recurrence. The positive, constant function $T: D \to \{1\}$ is a solution of the recurrence, i.e., T agrees with f on $D - I = \{1\}$ and satisfies

$$T(x) = \frac{1}{2} \cdot T\big(r(x)\big) + \frac{1}{2}$$

for all $x \in I$. Now let J be a non-empty upper subset of I, so J is an interval of positive length with $\sup J = 2$. (e.g., maybe $J = I$.) Define

$$S = \big(D, J, a, b, T|_{D-J}, g|_J, h|_J\big).$$

Lemma 20.6 implies S is an admissible recurrence with T as a solution. (Furthermore, S is proper by Lemma 9.7.) Let d be the depth-of-recursion function for S. Observe that $r^{-1}(x) \in (x, 2) \subset J$ for all $x \in J$. By an obvious inductive argument, we conclude that

$$r^{-n}(x) \in J$$

and

$$d\big(r^{-n}(x)\big) = d(x) + n > n$$

for all $x \in J$ and all $n \in Z^+$. (The exponent $-n$ refers to composition of functions, not powers of function values.) Therefore, S does not have bounded depth of recursion on the bounded interval J. Thus S violates the bounded depth and ratio conditions (see Lemma 9.6).

Define an equivalence relation $\sim$ on I by $y \sim z$ when $r^m(y) = z$ for some integer m, so each equivalence class is countably infinite. Let P be the set of equivalence classes, so P is a partition of I. By the axiom of choice, there exists a transversal L of $\sim$. Observe that $|L| = |P|$, so

$$\max(|L|, |\boldsymbol{N}|) = \max(|P|, |\boldsymbol{N}|) = |I| = |\boldsymbol{R}|.$$

We conclude from $|\boldsymbol{N}| < |\boldsymbol{R}|$ that $|L| = |\boldsymbol{R}|$. There exist infinitely many bijections $\lambda: L \to \boldsymbol{R}$. Each such λ can be uniquely extended to a solution T_λ of R. Observe that

$$\boldsymbol{R} = range(\lambda) \subseteq range(T_\lambda) \subseteq \boldsymbol{R},$$

so $range(T_\lambda) = \boldsymbol{R}$. In particular, the solution T_λ is unbounded above and below. The domain, D, of T_λ is bounded, so T_λ is neither $\Theta(1)$ nor locally $\Theta(1)$. Distinct bijections from L onto $\boldsymbol{R}$ determine distinct solutions of R, so there are infinitely many solutions of R that are surjections onto $\boldsymbol{R}$ and are neither $\Theta(1)$ nor locally $\Theta(1)$.

A finitely recursive, proper admissible recurrence *R* with a bounded recursion set, and a positive constant solution *T* such that the recurrence *S* of Lemma 20.6 does not satisfy the bounded depth condition for any choice of *J*. Define $D = [1, 3)$ and $I = [2, 3)$. Define the increasing sequence $t_0, t_1, t_2, \ldots$ by $t_0 = 1$ and

$$t_j = 3 - \frac{1}{j}$$

for $j > 0$. Then $t_1 = 2$ and

$$\lim_{j \to \infty} t_j = 3.$$

Let $Y_j = [t_j, t_{j+1})$ for each non-negative integer j, so $Y_0 = [1, 2) = D - I$ and I is the disjoint union of $Y_1, Y_2, Y_3, \ldots$ For each positive integer j, define the bijection $\varphi_j: Y_j \to Y_{j-1}$ by

$$\varphi_j(x) = t_{j-1} + \left(\frac{x - t_j}{t_{j+1} - t_j}\right)(t_j - t_{j-1}).$$

Define $r: I \to D$ by $r|_{Y_j} = \varphi_j$ for each positive integer j. Let $a = b = 1/2$. Define functions $f: D - I \to \{1\}$ and $g: I \to \{1/2\}$. Define $h: I \to \boldsymbol{R}$ by $h(t) = r(t) - bt$. The admissible recurrence $R = (D, I, a, b, f, g, h)$, i.e., the recurrence

$$T(x) = \begin{cases} 1, & \text{for } x \in D - I \\ \frac{1}{2} \cdot T(r(x)) + \frac{1}{2}, & \text{for } x \in I, \end{cases}$$

is proper and finitely recursive with the unique solution $T(x) = 1$ for all $x \in D$. Let J be any non-empty upper subset of I, and let S be the corresponding admissible recurrence of Lemma 20.6 with J as its recursion set. Let d be the depth-of-recursion function for S. Let $z \in J$, so $z \in Y_\alpha$ for some integer $\alpha > 0$. Then $Y_{\alpha+n} \subset J$ for each integer $n > 0$.

Furthermore, $d(J) \geq d(Y_{\alpha+n}) = d(Y_\alpha) + n > d(z) + n > n$ for each such n. Therefore, $d(J) = \infty$. J is bounded, so the bounded depth condition is violated by S.

Definition. Let k be a positive integer. A $(k+3)$-tuple $(x_0, b_1, \ldots, b_k, p, \varepsilon)$ of real numbers satisfies the *technical condition* if

(1) $0 < b_i < 1$ for all $i \in \{1, \ldots, k\}$,

(2) $\varepsilon > 0$,

(3) $x_0 > e^{1/b_i}$ for all $i \in \{1, \ldots, k\}$,

(4) $\log^{\varepsilon/2} x_0 > 2$, and

(5) for all $x \geq x_0$ and all $i \in \{1, \ldots, k\}$,

(a) $\left(1 - \frac{1}{b_i \log^{1+\varepsilon} x}\right)^{|p|} \left(1 + \frac{1}{\log^{\varepsilon/2}\left(b_i x + \frac{x}{\log^{1+\varepsilon} x}\right)}\right) > 1 + \frac{1}{\log^{\varepsilon/2} x}$,

(b) $\left(1 + \frac{1}{b_i \log^{1+\varepsilon} x}\right)^{|p|} \left(1 - \frac{1}{\log^{\varepsilon/2}\left(b_i x + \frac{x}{\log^{1+\varepsilon} x}\right)}\right) < 1 - \frac{1}{\log^{\varepsilon/2} x}$.

Remarks about the technical condition. We include (2) for convenience although it is a consequence of (1), (3), and (4). If $(x_0, b_1, \ldots, b_k, p, \varepsilon)$ satisfies the technical condition, then $(y, b_1, \ldots, b_k, p, \varepsilon)$ satisfies the technical condition for all $y \geq x_0$. Parts (1) and (3) imply $x_0 > e$. Let $i \in \{1, \ldots, k\}$. Part (3) and positivity of b_i (by (1)) imply

$$b_i x_0 > b_i e^{1/b_i} > b_i \frac{1}{b_i} = 1,$$

which is a strict version of the inequality $x_0 \geq 1/b_i$ assumed by Theorems 1 and 2 of [Le]. Parts (5a) and (5b) are strict versions of conditions 4(a) and 4(b) of Theorem 2 of [Le] with p replaced by $|p|$. Sections 24 and 25 explain the main reason we use $|p|$. The replacement of p by $|p|$ in (5a) also combines with (1), (2), (3), and strictness of (5a) to imply that all admissible recurrences satisfying the *modified Leighton hypothesis* also satisfy the ratio condition (see the proof of Lemma 20.9). Part (4) of the technical condition is a strict version of condition 4(d) of [Le] and implies a strict version of condition 4(c) of [Le].

As explained in Section 19, parts (1), (2), and (5a) of the technical condition along with $x_0 > 1/b_i$ (the bound $x_0 \geq 1/b_i$ is sufficient) can be construed to imply $b_i \log^{1+\varepsilon} x_0 > 1$ if p is nonzero or we follow the convention that 0^0 is undefined. The technical condition includes (3) partly to make explicit our intention that $b_i \log^{1+\varepsilon} x_0 > 1$. Parts (1), (2), and (3) imply

$$b_i \log^{1+\varepsilon} x_0 > b_i \left(\frac{1}{b_i}\right)^{1+\varepsilon} = \left(\frac{1}{b_i}\right)^{\varepsilon} > 1^{\varepsilon} = 1.$$

All denominators that appear in (5a) and (5b) are defined and positive and in particular are non-zero: Let $x \geq x_0$, so $x > 1$, i.e., $\log x > 0$, so the denominators $\log^{\varepsilon/2} x$ and $\log^{1+\varepsilon} x$ are positive. Since $b_i > 0$, the denominator $b_i \log^{1+\varepsilon} x$ is also positive. Let

$$v = b_i x + \frac{x}{\log^{1+\varepsilon} x}.$$

Then

$$v > b_i x \geq b_i x_0 > 1,$$

so $\log(v) > 0$, which implies the denominator $\log^{\varepsilon/2} v$ is positive.

Parts (1) and (3) of the technical condition are used in the proof of Lemma 25.1, which says $b_i x_0 > e$, so $v > e$ and

$$\log^{\varepsilon/2} v > 1.$$

The proof of Lemma 25.2 uses this improved bound for the denominator $\log^{\varepsilon/2} v$ in conjunction with condition (5b).

Lemma 20.7. If

(1) k is a positive integer,

(2) $b_1, \ldots, b_k$ are real numbers such that $0 < b_i < 1$ for each i,

(3) p is a real number, and

(4) $\varepsilon > 0$,

then there exists a real number x_0 such that $(x_0, b_1, \ldots, b_k, p, \varepsilon)$ satisfies the technical condition.

We postpone our proof of Lemma 20.7 to Sections 27 and 28. No proof is given in [Le] of the corresponding assertion about Condition (4) of Leighton's Theorem 2. However, a footnote says "Such a constant value of x_0 can be shown to exist using standard Taylor series expansions and asymptotic analysis."

Definition. An admissible recurrence

$$R = (D, I, a_1, \ldots, a_k, b_1, \ldots, b_k, f, g, h_1, \ldots, h_k)$$

satisfies the *modified Leighton hypothesis* relative to $\varepsilon > 0$ if R satisfies Leighton's noise condition on I relative to ε, and

$$(\inf I, b_1, \ldots, b_k, p, \varepsilon)$$

satisfies the technical condition where p is the Akra-Bazzi exponent of R.

By definition of Leighton's noise condition, the recursion set of an admissible recurrence that satisfies the modified Leighton hypothesis must be contained in the interval $(1, \infty)$.

Examples of proper admissible recurrences that satisfy the hypothesis of Leighton's Theorem 2 for some $\varepsilon > 0$ but violate (only) parts (5a) and (5b) of the associated technical conditions. In particular, they violate the modified Leighton hypothesis relative to ε. Let

$$\delta = \frac{\log 100}{\log \log 10000}.$$

The (proper when $x_0 = 10000$) admissible recurrences described in Sections 13 and 15 satisfy the hypothesis of Leighton's Theorem 2 with $\varepsilon = 0.74$ and $\varepsilon = \delta - 1 \approx 1.074$, respectively. They also satisfy the first four parts of the associated technical conditions. Both recurrences have $b_1 = 0.99$. The recurrences of Sections 13 and 15 have Akra-Bazzi exponents $p = -1$ and $p = 0$, respectively. Recall from Section 15 that $\log^\delta 10000 = 100$. Parts (5a) and (5b) of the technical condition are violated by both recurrences for $x = 10000$ because

$$b_1 + \frac{1}{\log^{1+\varepsilon} x} \geq 0.99 + \frac{1}{\log^\delta 10000} = 1.$$

The recurrence from Section 15 satisfies conditions (5a) and (5b) for all $x > 10000$. (Recall that 10000 is the infimum of the recursion set.) The same recurrence would satisfy the modified Leighton hypothesis relative to $\varepsilon = \delta - 1$ if conditions (5a) and (5b) were modified to be non-strict inequalities.

The critical next lemma is similar to (the incorrect) Theorem 2 of [Le] and will be established in Section 26 as a consequence of Lemma 26.1, which is proved using an adaptation of arguments in [Le]. The proposition is applied in the proof of our main result, Theorem 20.11.

Lemma 20.8. Let $\varepsilon > 0$. If R is an admissible recurrence that satisfies the modified Leighton hypothesis relative to ε, then R has a unique solution T, which satisfies the strong Akra-Bazzi condition relative to R and g for each tame extension g of the incremental cost of R.

The next proposition is used in the proof of Lemma 26.1.

Lemma 20.9. Let $\varepsilon > 0$. If

$$R = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k)$$

is an admissible recurrence that satisfies the modified Leighton hypothesis relative to ε, then R satisfies the bounded depth and strong ratio conditions and has a unique solution, which is locally $\Theta(1)$.

Proof. Define $x_0 = \inf I$ and let p be the Akra-Bazzi exponent for R, so the $(k+3)$-tuple

$$(x_0, b_1, \dots, b_k, p, \varepsilon)$$

satisfies the technical condition. (Parts (1) and (3) of that condition imply $x_0 > 1$, so I is contained in $(1, \infty)$ as required for satisfaction of Leighton's noise condition on I relative to ε.) For all $i \in \{1, \dots, k\}$, define

$$\beta_i = \sup_{x \in I} \left(b_i + \frac{h_i(x)}{x} \right).$$

Satisfaction of Leighton's noise condition on I implies

$$\beta_i \leq b_i + \sup_{x \in I} \left(\frac{1}{\log^{1+\varepsilon} x} \right) = b_i + \frac{1}{\log^{1+\varepsilon} x_0}$$

for all $x \in I$ and each index i. As previously explained, parts (1), (2), and (3) of the technical condition imply

$$b_i \log^{1+\varepsilon} x_0 > 1,$$

so that

$$\left(1 - \frac{1}{b_i \log^{1+\varepsilon} x_0} \right)^{|p|} \leq 1$$

(with equality if $p = 0$). (If $|p|$ is replaced with p as in [Le], the inequality above is reversed when $p < 0$). The inequality above combines with part (5a) of the technical condition to imply

$$\log^{\varepsilon/2} \left(b_i x_0 + \frac{x_0}{\log^{1+\varepsilon} x_0} \right) < \log^{\varepsilon/2} x_0,$$

so that

$$b_i + \frac{1}{\log^{1+\varepsilon} x_0} < 1,$$

which implies $\beta_i < 1$. Define

$$\beta = \sup_{1 \leq i \leq k} \beta_i,$$

so that $\beta < 1$ and

$$b_i x + h_i(x) \leq \beta_i x \leq \beta x$$

for all $x \in I$ and each index i. Thus R satisfies the ratio condition. Lemma 9.6 implies R satisfies the bounded depth condition and has a unique solution, which is locally $\Theta(1)$. Now define

$$\alpha_i = \inf_{x \in I}\left(b_i + \frac{h_i(x)}{x}\right)$$

for each index i. Satisfaction of Leighton's noise condition on I relative to ε implies

$$\alpha_i \geq b_i - \sup_{x \in I}\left(\frac{|h_i(x)|}{x}\right) \geq b_i - \sup_{x \in I}\left(\frac{1}{\log^{1+\varepsilon} x}\right) = b_i - \frac{1}{\log^{1+\varepsilon} x_0} = \frac{b_i \log^{1+\varepsilon} x_0 - 1}{\log^{1+\varepsilon} x_0}.$$

We conclude from $\log x_0 > 0$ and $b_i \log^{1+\varepsilon} x_0 > 1$ that $\alpha_i > 0$. Define

$$\alpha = \inf_{1 \leq i \leq k} \alpha_i ,$$

so that $\alpha > 0$ and

$$\alpha x \leq \alpha_i x \leq b_i x + h_i(x)$$

for all $x \in I$ and each index i. Since R satisfies the ratio condition, it also satisfies the strong ratio condition. □

The next lemma essentially reduces our study of locally $\Theta(1)$ solutions of admissible recurrences to solutions of admissible recurrences that satisfy the modified Leighton hypothesis.

Lemma 20.10. Suppose

$$R = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k)$$

is an admissible recurrence with unbounded recursion set I. If T is a locally $\Theta(1)$ solution of R, then there exists a non-empty upper subset J of I such that the admissible recurrence

$$S = \left(D, J, a_1, \dots, a_k, b_1, \dots, b_k, T|_{D-J}, g|_J, h_1|_J, \dots, h_k|_J\right)$$

has T as its only solution and satisfies the modified Leighton hypothesis relative to some $\varepsilon > 0$.

Proof. Lemma 20.1 implies R satisfies Leighton's noise condition on some non-empty upper subset U of I relative to some $\varepsilon > 0$. Let p be the Akra-Bazzi exponent of R. By Lemma 20.7, there exists a real number z such that

$$(z, b_1, \dots, b_k, p, \varepsilon)$$

satisfies the technical condition. There exists $x_0 \in U$ such that $x_0 > z$ because

$$\sup U = \sup I = \infty.$$

Then

$$(x_0, b_1, \ldots, b_k, p, \varepsilon)$$

also satisfies the technical condition. In particular, $x_0 > 1$.

Define the upper subset $J = I \cap [x_0, \infty)$ of I, so that J contains x_0 and is therefore non-empty. Let

$$S = \left(D, J, a_1, \ldots, a_k, b_1, \ldots, b_k, T|_{D-J}, g|_J, h_1|_J, \ldots, h_k|_J\right).$$

Lemma 20.6 implies S is indeed an admissible recurrence with T as a solution. The set J is contained in U because J and U are upper subsets of I with $\min J = x_0 \in U$. Therefore, the recurrence S satisifes Leighton's noise condition on its recursion set J relative to ε. Since $\inf J = \min J = x_0 > z$, we know that

$$(\inf J, b_1, \ldots, b_k, p, \varepsilon)$$

satisfies the technical condition. Observe that p is the Akra-Bazzi exponent for S as well as for R. Therefore, S satisfies the modified Leighton hypothesis relative to ε. Lemma 20.9 implies S has a unique solution, which must be T. □

Lemma 20.9 would be unnecessary if the modified Leighton Hypothesis redundantly assumed the strong ratio condition. Lemma 9.6 would imply the other assertions of Lemma 20.9: satisfaction of the bounded depth condition and existence of a unique solution, which is locally $\Theta(1)$. In the proof of Lemma 20.10, Corollary 9.9 and Lemma 9.7 would allow us (with a little finesse) to choose the recurrence S to satisfy the strong ratio condition and have a unique solution, which must be T.

However, we intentionally omit the strong ratio condition from the modified Leighton Hypothesis to maintain the analogy between our definition and the hypothesis of Theorem 2 in [Le]. Furthermore, Lemma 20.9 is of some interest in itself.

The theorem below and its two corollaries replace Theorem 2 of [Le].

Theorem 20.11. Suppose T is a solution of an admissible recurrence R. Let G be the set of tame extensions of the incremental cost of R. Either all or none of the following statements are true:

(1) T is locally $\Theta(1)$.

(2) T satisfies the strong Akra-Bazzi condition relative to R and g for *some* $g \in G$.

(3) T satisfies the strong Akra-Bazzi condition relative to R and g for *all* $g \in G$.

Proof. By definition of an admissible recurrence, G is non-empty, so (3) implies (2). By Corollary 20.4, part (2) implies (1). We now show that (1) implies (3). Suppose T is locally $\Theta(1)$, and let $g \in G$.

Let I be the recursion set of R, and let $A: I \to \boldsymbol{R}^+$ be the Akra-Bazzi estimate for R relative to g. Lemma 20.2 implies A is locally $\Theta(1)$.

Suppose I is bounded, so there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \boldsymbol{R}^+$ such that

$$\alpha_1 \leq T(x) \leq \alpha_2,$$

$$\beta_1 \leq A(x) \leq \beta_2,$$

and

$$\frac{\alpha_1}{\beta_2} A(x) \leq T(x) \leq \frac{\alpha_2}{\beta_1} A(x)$$

for all $x \in I$. Then T satisfies the strong Akra-Bazzi condition relative to R and g. We now assume instead that I is unbounded. Let

$$R = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, \gamma, h_1, \dots, h_k).$$

Lemma 20.10 implies there exists a non-empty upper subset J of I such that the admissible recurrence

$$S = \left(D, J, a_1, \dots, a_k, b_1, \dots, b_k, T|_{D-J}, \gamma|_J, h_1|_J, \dots, h_k|_J\right)$$

has T as its only solution and satisfies the modified Leighton hypothesis relative to some $\varepsilon > 0$. We conclude from $\gamma = g|_I$ that $\gamma|_J = g|_J$. Lemma 20.8 implies T satisfies the strong Akra-Bazzi condition relative to S and g. Lemma 20.6 implies T also satisfies the strong Akra-Bazzi condition relative to R and g. □

Corollary 20.12. If R is an admissible recurrence that satisfies the bounded depth condition, then R has a unique solution T, which is locally $\Theta(1)$ and satisfies the strong Akra-Bazzi condition relative to R and each tame extension of the incremental cost of R.

Proof. Lemma 9.10 implies R has a unique solution T, which is locally $\Theta(1)$. The proposition follows from Theorem 20.11. □

Corollary 20.13. If R is an admissible recurrence that satisfies the ratio condition, then R has a unique solution T, which is locally $\Theta(1)$ and satisfies the strong Akra-Bazzi condition relative to R and each tame extension of the incremental cost of R.

Proof. Lemma 9.6 implies R satisfies the bounded depth condition. The proposition follows from Corollary 20.12. □

Independence of strong Akra-Bazzi condition from choice of recurrence. By Theorem 20.11, satisfaction of the strong Akra-Bazzi condition by a solution T of an admissible recurrence is independent of the choice of admissible recurrence that has T as a solution (or any specific tame extension of the incremental cost).

Violation of Akra-Bazzi formula by recurrence with zero in closure of recursion set. We now illustrate one reason the definition of a semi-divide-and-conquer recurrence requires the recursion set to have a positive lower bound. Let

$$R = (D, I, a, b, f, g, h)$$

where $D = [0, \infty)$, $I = (0, \infty)$, $a = 2$, $b = 1/2$, $f: \{0\} \to \{1\}$, $g: I \to \{1\}$, and $h: I \to \boldsymbol{R}$ is defined by

$$h(x) = \begin{cases} -x/2, & \text{for } 0 < x \leq 1 \\ 0, & \text{for } x > 1. \end{cases}$$

Observe that $\inf I = 0$, so R violates condition (2) of the definition of a semi-divide-and-conquer recurrence and is therefore inadmissible. Define $p = 1$, so $ab^p = 1$. Notice that

$$x/2 + h(x) = \begin{cases} 0, & \text{for } 0 < x \leq 1 \\ x/2, & \text{for } x > 1. \end{cases}$$

The recurrence described by R, i.e.,

$$T(x) = \begin{cases} 1, & \text{for } x = 0 \\ 2T\big(x/2 + h(x)\big) + 1, & \text{for } x > 0, \end{cases}$$

is finitely recursive and therefore has a unique solution T by Lemma 8.2. Let d be the depth-of-recursion function for the recurrence, so

$$d(x) = \begin{cases} 0, & \text{for } x = 0 \\ 1, & \text{for } 0 < x \leq 1 \\ \lceil \log_2 x \rceil + 1, & \text{for } x > 1. \end{cases}$$

An easy inductive argument on $d(x)$ shows that

$$T(x) = 2^{d(x)+1} - 1$$

for all $x \geq 0$. (The base case of the induction is $T(0) = 1 = 2^{0+1} - 1$.) For all $x > 1$, we have

$$\log_2 x + 1 \leq d(x) < \log_2 x + 2$$

and

$$3x < 4x - 1 = 2^{\log_2 x + 2} - 1 \leq T(x) < 2^{\log_2 x + 3} - 1 = 8x - 1 < 8x.$$

Therefore, $T(x) = \Theta(x)$. Observe that $T(x) \leq 3$ when $0 \leq x \leq 1$, so

$$1 \le T(x) < 8x + 3$$

for all $x \in D$, which implies T is locally $\Theta(1)$. However, Theorem 20.11 is inapplicable to the inadmissible recurrence. The Akra-Bazzi integral (with our modified lower limit of integration) diverges for all $x > 0$:

$$\int_0^x \frac{g(u)}{u^{p+1}}\,du = \lim_{t \to 0^+} \int_t^x \frac{1}{u^2}\,du = \lim_{t \to 0^+} \left(\frac{1}{t} - \frac{1}{x}\right) = \infty.$$

The only plausible interpretation of our modified Akra-Bazzi formula

$$T(x) = \Theta\left(x^p\left(1 + \int_0^x \frac{g(u)}{u^{p+1}}\,du\right)\right)$$

is the false conclusion that $T(x) = \infty$ for sufficiently large x. The Akra-Bazzi formula is unsatisfied by our example.

21. Integer Divide-and-Conquer Recurrences

Recurrences encountered in computer science are commonly defined on the positive integers or the non-negative integers. Restriction of attention to such recurrences avoids many complications. Theorem 21.2 is our main result about recurrences with recursion sets consisting of integers.

Lemma 21.1. If R is a divide-and-conquer recurrence with a recursion set that contains only integers, then R satisfies the bounded depth condition and has a unique solution, which is locally $\Theta(1)$.

Proof. Let I be the recursion set of R, so I is positive (i.e., $I \subseteq \boldsymbol{Z}^{+}$) by definition of a divide-and-conquer recurrence. Let

$$A_j = I \cap (-\infty, j]$$

for each non-negative integer j, so

$$I \cap (-\infty, j+1) = A_j$$

for each such j. Observe that $A_0 = \emptyset$. Let d be the depth-of-recursion function for R, so

$$d(A_0) = d(\emptyset) = 0.$$

Suppose m is a non-negative integer with $d(A_m) \leq m$. The recurrences R is proper by hypothesis, so

$$r(A_{m+1}) \subseteq D \cap (-\infty, m+1) \subseteq A_m \cup (D \backslash I)$$

for each dependency $r: I \rightarrow D$ of R. Then

$$d(A_{m+1}) \leq d\big(A_m \cup (D \backslash I)\big) + 1 = \max\{d(A_m), d(D \backslash I)\} + 1 \leq max\{m, 0\} + 1$$

$$= m + 1.$$

By induction, $d\big(A_j\big) \leq j$ for each non-negative integer j.

If X is a bounded subset of I, then $X \subseteq A_n$ for some non-negative integer n, so

$$d(X) \leq d(A_n) \leq n < \infty.$$

Lemma 9.2 implies R satisfies the bounded depth condition. In particular, R is finitely recursive, so Corollary 8.5 implies R has a unique solution, T, which is positive.

Let S be a bounded subset of I, so S is a bounded set of integers, which implies S is a finite set. Then $T(S)$ is a finite set of positive real numbers, so $T(S)$ has minimum and maximum elements, which are positive real numbers, so T is $\Theta(1)$ on S. Therefore, the restriction of T to I is locally $\Theta(1)$. Lemma 9.1 implies T is locally $\Theta(1)$. □

Theorem 21.2. Let R be a divide-and-conquer recurrence with low noise. If the recursion set of R contains only integers and the incremental cost of R has polynomial growth, then R is admissible and has a unique solution, which satisfies the strong Akra-Bazzi condition relative to R and each tame extension of the incremental cost of R.

Proof. Lemma 21.1 implies R satisfies the bounded depth condition and has a unique solution T, which is locally $\Theta(1)$. Corollary 5.3 implies the incremental cost of R has a continuous, polynomial-growth extension to $\boldsymbol{R}^+$. Continuity of the extension implies it is locally Riemann integrable and is therefore tame. Now R satisfies the definition of an admissible recurrence. The proposition follows from Theorem 20.11 (or Corollary 20.12). □

Base case of integer recurrence. Let R be a divide-and-conquer recurrence with domain D, recursion set I, and base case f. Suppose D has a finite lower bound (as is common in practice), so the domain $D\backslash I$ of f has the same finite lower bound. The recursion set I is a non-empty upper subset of D, so $D\backslash I$ has a finite upper bound. Thus $D\backslash I$ is a bounded set. Further suppose that $D\backslash I$ contains only integers (e.g., if D contains only integers). Then $D\backslash I$ is a bounded set of integers and is therefore finite. Our definition of a divide-and-conquer recurrence requires that f has a positive lower bound and a finite upper bound. We note (again) that a real-valued function on a finite set has a positive lower bound and finite upper bound if and only if the function is positive. We conclude that when $D\backslash I$ is a set of integers and $inf\ D > -\infty$, the requirement $f = \Theta(1)$ is equivalent to positivity of f.

Obvious tame extensions of incremental costs. Many polynomial-growth functions have obvious, natural tame extensions. For example, the function $x \mapsto \log x$ on $[2, \infty)$ is a tame extension of the polynomial-growth function $n \mapsto \log n$ on $\boldsymbol{Z} \cap [2, \infty)$.

Floors and Ceilings in Dependencies. Let $b \in (0,1)$. Functions of the form $x \mapsto \lfloor bx \rfloor$ or $x \mapsto \lceil bx \rceil$ on a positive set have $x \mapsto bx$ as a linear approximation. Observe that

$$|\lfloor bx \rfloor - bx| < 1 = O\left(\frac{x}{\log^c x}\right)$$

and

$$|\lceil bx \rceil - bx| < 1 = O\left(\frac{x}{\log^c x}\right)$$

for all real c (in particular, for *some* $c > 1$). Dependencies of the form $x \mapsto \lfloor bx \rfloor$ or $x \mapsto \lceil bx \rceil$ can be represented as

$$x \mapsto bx + (\lfloor bx \rfloor - bx)$$

or

$$x \mapsto bx + (\lceil bx \rceil - bx),$$

respectively, i.e. with noise terms $x \mapsto \lfloor bx \rfloor - bx$ or $x \mapsto \lceil bx \rceil - bx$, respectively. These representations satisfy the requirement of low noise for an admissible recurrence.

Example. The recurrence

$$T(n) = \begin{cases} 1, & \text{for } n = 1 \\ 2, & \text{for } n = 2 \\ 2T(\lfloor n/3 \rfloor) + T(n - 2\lfloor n/3 \rfloor) + n, & \text{for } n \geq 3 \end{cases}$$

with domain $\boldsymbol{Z}^+$ can be represented as a divide-and-conquer recurrence R described as

$$T(n) = \begin{cases} f(n), & \text{for } n \in \boldsymbol{Z}^+ \backslash I \\ 2T\big(n/3 + h_1(n)\big) + T\big(n/3 + h_2(n)\big) + n, & \text{for } n \in I \end{cases}$$

where $I = \boldsymbol{Z} \cap [3, \infty)$, $f: \{1,2\} \to \boldsymbol{R}$ is defined by $f(1) = 1$ and $f(2) = 2$, and $h_1, h_2: I \to \boldsymbol{R}$ are defined by

$$h_1(n) = \lfloor n/3 \rfloor - n/3$$

and

$$h_2(n) = 2(n/3 - \lfloor n/3 \rfloor),$$

respectively. The incremental cost is the function $g: I \to R$ defined by $g(n) = n$. Corollary 2.12 implies g has polynomial growth. The inequalities $|h_1(n)| < 1$ and $|h_2(n)| < 2$ imply

$$|h_i(n)| = O\left(\frac{n}{\log^c n}\right)$$

for each real number c and each $i \in \{1,2\}$. In particular, R has low noise. Theorem 21.2 implies R is an admissible recurrence and has a unique solution T, which satisfies the strong Akra-Bazzi condition relative to R and each tame extension of g.

Let g^* be the identity function, $x \mapsto x$, on $[3, \infty)$, so g^* is an extension of g. The function g^* has polynomial growth by Corollary 2.12 and is locally Riemann integrable. Therefore, g^* is a tame extension of g.

The Akra-Bazzi exponent is 1 because

$$\frac{2}{3^1} + \frac{1}{3^1} = 1.$$

Therefore,

$$T(n) = \Theta\left(n^1\left(1 + \int_3^n \frac{u}{u^{1+1}} du\right)\right) = \Theta\left(n\left(1 + \int_3^n \frac{du}{u}\right)\right) = \Theta(n \log n).$$

Lemma 21.1 implies R is finitely recursive, which implies T is integer-valued (by induction on the depth-of-recursion).

The ratio and strong ratio conditions play a role elsewhere, so we include the following proposition for sake of completeness:

Lemma 21.3. If R is a divide-and-conquer recurrence with positive domain, low noise and a recursion set that contains only integers, then R satisfies the strong ratio condition.

Proof. Let D and I be the domain and recursion set, respectively, of the recurrence. Let $r_1, \ldots, r_k : I \to D$ be the dependencies, which are positive functions because D is a positive set. For each non-empty, finite subset S of I, define

$$Q(S) = \{r_i(n)/n : n \in S \text{ and } 1 \le i \le k\}.$$

Positivity of I implies the denominators appearing in the definition of $Q(S)$ are non-zero. The elements of $Q(S)$ are positive because I is a positive set and $r_1, \ldots, r_k$ are positive functions. The non-empty set $Q(S)$ of positive real numbers is finite and therefore has a minimum and maximum, which are positive. Furthermore, $q < 1$ for all $q \in Q(S)$ because R is proper. Therefore, $\max Q(S) < 1$.

If I is finite, then $0 < \min Q(I) \le \max Q(I) < 1$, i.e, the strong ratio condition is satisfied. Now suppose I is infinite. We conclude from $I \subseteq \boldsymbol{Z}$ that I is unbounded. Lemma 9.8 and low noise of R imply there exists a non-empty upper subset J of I and real numbers α and β with $0 < \alpha \le \beta < 1$ such that $\alpha m \le r_i(m) \le \beta m$ for all $m \in J$ and each index i. If $I = J$, then R satisfies the strong ratio condition. Suppose $I \ne J$, so $I \setminus J \ne \emptyset$. The set $I \setminus J$ is a set of positive integers bounded above by $\min J$, so $I \setminus J$ is finite. Let L be the minimum of α and $\min Q(I \setminus J)$, and let U be the maximum of β and $\max Q(I \setminus J)$. Then $0 < L \le U < 1$ and $Ln \le r_i(n) \le Un$ for all $n \in I$ and each index i. In other words, R satisfies the strong ratio condition. □

A proper, admissible recurrence with domain $\boldsymbol{N}$ that satisfies the ratio condition but does not satisfy the strong ratio condition. The proper, admissible recurrence

$$T(n) = \begin{cases} 1, & \text{for } n \in \{0,1\} \\ 2T(\lfloor n/2 \rfloor - 1) + 1, & \text{for } n \in I \end{cases}$$

with domain $\boldsymbol{N}$ and recursion set $I = \boldsymbol{Z} \cap [2, \infty)$ satisfies the ratio condition because

$$\lfloor n/2 \rfloor - 1 < n/2$$

for all $n \in I$. However,

$$\lfloor 2/2 \rfloor - 1 = 0$$

and

$$\lfloor 3/2 \rfloor - 1 = 0,$$

so there is no positive, linear lower bound for the dependency $n \mapsto \lfloor n/2 \rfloor - 1$. Therefore, the strong ratio condition is violated.

Example of a divide-and-conquer recurrence with domain $\boldsymbol{Z}^+$ that does not have low noise and does not satisfy either bound of the strong ratio condition. The recurrence

$$T(n) = \begin{cases} 1, & \text{for } n \in \{1,2\} \\ T(n-1) + 1, & \text{for odd } n \geq 3 \\ T(\lfloor n/\log n \rfloor) + 1, & \text{for even } n \geq 4 \end{cases}$$

with domain $\boldsymbol{Z}^+$ can be represented as the divide-and-conquer recurrence

$$T(n) = \begin{cases} 1, & \text{for } n \in \{1,2\} \\ T\big(n/2 + h(n)\big) + 1, & \text{for } n \in I \end{cases}$$

with domain $\boldsymbol{Z}^+$ and recursion set $I = \boldsymbol{Z} \cap [3, \infty)$ where $h: I \to \boldsymbol{R}$ is defined by

$$h(n) = \begin{cases} n/2 - 1, & \text{if } n \text{ is odd} \\ \lfloor n/\log n \rfloor - n/2\,, & \text{if } n \text{ is even.} \end{cases}$$

The recurrence does not have low noise. The ratio condition is also violated. Furthermore, there is no positive linear lower bound as required by the strong ratio condition.

22. Replacement for Leighton's Lemma 2

The following proposition is our replacement for Lemma 2 of [Le].

Lemma 22.1. Let R be a divide-and-conquer recurrence with recursion set I and incremental cost g, and let p be a real number. Suppose R satisfies the strong ratio condition and g has a tame extension G. Let S be the set of dependencies of R and let

$$I^* = \{x \in I : r(x) \in domain(G) \text{ for all } r \in S\}.$$

Then there exist positive real numbers c_1 and c_2 such that

$$c_1 g(x) \leq x^p \int_{r(x)}^{x} \frac{G(u)}{u^{p+1}} du \leq c_2 g(x)$$

for all $x \in I^*$ and all $r \in S$.

Proof. If I^* is empty, the proposition is vacuously satisfied for every choice of positive real numbers c_1 and c_2. Therefore, we may assume that I^* is non-empty.

Let $H = domain(G)$. By definition of a tame function, H is a non-empty, positive interval. Define $f: H \to \boldsymbol{R}$ by

$$f(u) = \frac{G(u)}{\delta(u)}$$

for all $u \in H$ where $\delta: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ is defined by $\delta(w) = w^{p+1}$ for all $w \in \boldsymbol{R}^+$. (The function δ is introduced as a minor convenience; our reason for defining δ on $\boldsymbol{R}^+$ instead of H will be explained later.) The function f is tame by Corollary 10.3. In particular, f is locally Riemann integrable. If G is identically zero, then f and g are also identically zero and the proposition is satisfied with $c_1 = c_2 = 1$. Therefore, we may assume G is not identically zero. Lemma 2.7 (also Lemma 10.1(1)) implies G is positive, so f and g are positive.

Satisfaction of the strong ratio condition implies the existence of real numbers α and β such that

$$0 < \alpha \leq \beta < 1$$

and

$$\alpha t \le \varphi(t) \le \beta t$$

for all $t \in I$ and all $\varphi \in S$. Positivity of G and H implies $\Psi_{1/\alpha}(G)$ and $\Psi_{1/\beta}(G)$ are defined; they are positive by Lemma 2.10(2) and finite by Lemma 2.16.

Define positive real numbers

$$c_1 = \frac{(1-\beta)}{\Psi_{1/\beta}(G) \cdot \max(1, \beta^{p+1})}$$

and

$$c_2 = \frac{(1-\alpha)\Psi_{1/\alpha}(G)}{\min(1, \alpha^{p+1})}.$$

Recall that $I = domain(g)$ by definition of a semi-divide-and-conquer recurrence. Then $I \subseteq H$ because G is an extension of g. Consequently, $I^* \subseteq H$.

Suppose $x \in I^*$, so $x \in H$ and $x > 0$. Define $q: S \to H$ by $q(\lambda) = \lambda(x)$ for all $\lambda \in S$. The set S is finite and non-empty by definition of a semi-divide-and-conquer recurrence, so $q(S)$, i.e.,

$$\{\lambda(x) : \lambda \in S\},$$

is a non-empty, finite set of real numbers and must therefore have a minimum element, z, i.e.,

$$z = \min_{\lambda \in S} \lambda(x).$$

In particular, $z = \mu(x)$ for some $\mu \in S$. Then $z \in H$ and

$$0 < \alpha x \le z \le \lambda(x) \le \beta x < x$$

for all $\lambda \in S$. Connectivity of H combines with $z, x \in H$ and the inequalities above to imply

$$[L, x] \subseteq [z, x] \subseteq H \subseteq \boldsymbol{R}^+ = domain(\delta)$$

for all

$$L \in \{\beta x\} \cup \{\lambda(x) : \lambda \in S\}.$$

In particular, f is Riemann integrable on $[L, x]$ for each such L.

Let $r \in S$, so

$$[\beta x, x] \subseteq [r(x), x] \subseteq [z, x].$$

Non-negativity of f and x^p implies

$$x^p \int_{\beta x}^{x} f(u)du \;\le\; x^p \int_{r(x)}^{x} f(u)du \;\le\; x^p \int_{z}^{x} f(u)du.$$

We will show that

$$x^p \int_{\beta x}^{x} f(u)du \geq c_1 g(x)$$

and

$$x^p \int_{z}^{x} f(u)du \leq c_2 g(x),$$

so

$$c_1 g(x) \;\leq\; x^p \int_{r(x)}^{x} f(u)du \;\leq\; c_2 g(x)$$

as required. Observe that

$$\Lambda([\beta x, x]) = 1/\beta.$$

Then Lemma 2.10(4) and $x \in [\beta x, x] \subseteq H$ imply

$$\inf G([\beta x, x]) \geq \frac{G(x)}{\Psi_{1/\beta}(G)} = \frac{g(x)}{\Psi_{1/\beta}(G)}.$$

Monotonicity of the function δ combines with

$$[\beta x, x] \subseteq \boldsymbol{R}^+ = domain(\delta)$$

to imply

$$\max \delta([\beta x, x]) = \max\big(\delta(x), \delta(\beta x)\big) = \max(x^{p+1}, (\beta x)^{p+1}) = x^{p+1} \cdot \max(1, \beta^{p+1}).$$

Positivity of G and δ implies

$$\inf f([\beta x, x]) \;\geq\; \frac{\inf G([\beta x, x])}{\max \delta([\beta x, x])} \;\geq\; \frac{g(x)}{\Psi_{1/\beta}(G) \cdot x^{p+1} \cdot \max(1, \beta^{p+1})}.$$

Then positivity of x^p and $x - \beta x$ imply

$$x^p \int_{\beta x}^{x} f(u)du \;\geq\; x^p(x - \beta x) \cdot \inf f([\beta x, x]) \;\geq\; \frac{x^{p+1}(1-\beta)g(x)}{\Psi_{1/\beta}(G) \cdot x^{p+1} \cdot \max(1, \beta^{p+1})}$$

$$= \; c_1 g(x)$$

as claimed. Lemma 2.9(5) and $[z, x] \subseteq [\alpha x, x]$ imply

$$\Lambda([z, x]) \leq \Lambda([\alpha x, x]) = 1/\alpha.$$

Then Lemma 2.10(4) and $x \in [z, x] \subseteq H$ imply

$$\sup G([z, x]) \leq \Psi_{1/\alpha}(G) G(x) = \Psi_{1/\alpha}(G) g(x).$$

Monotonicity of the function δ combines with

$$[z, x] \subseteq [\alpha x, x] \subseteq \boldsymbol{R}^+ = domain(\delta)$$

to imply

$$\min \delta([z, x]) \geq \min \delta([\alpha x, x]) = \min\big(\delta(x), \delta(\alpha x)\big)$$

$$= \min(x^{p+1}, (\alpha x)^{p+1}) = x^{p+1} \cdot \min(1, \alpha^{p+1}).$$

(We defined $domain(\delta) = \boldsymbol{R}^+$ to ensure that $[\alpha x, x] \subseteq domain(\delta)$ as required by our argument above about $\min \delta([z, x])$.)

Positivity of G and δ implies

$$0 \ < \ \sup f([z, x]) \ \leq \ \frac{\sup G([z, x])}{\min \delta([z, x])} \leq \frac{\sup G([z, x])}{\min \delta([\alpha x, x])} \leq \frac{\Psi_{1/\alpha}(G) g(x)}{x^{p+1} \cdot \min(1, \alpha^{p+1})}.$$

Then the inequalities $x^p > 0$ and

$$\alpha x \leq z < x$$

imply

$$x^p \int_z^x f(u) du \ \leq \ x^p (x - z) \cdot \sup f([z, x]) \ \leq \ \frac{x^p (x - \alpha x) \Psi_{1/\alpha}(G) g(x)}{x^{p+1} \cdot \min(1, \alpha^{p+1})}$$

$$= \ c_2 g(x)$$

as claimed. The proposition is proved. □

Dependency on tame extension. The choice of tame extension G determines the valid choices for c_1 and c_2 in Lemma 22.1.

Example with empty I^*. The divide-and-conquer recurrence

$$T(u) = \begin{cases} 1, & \text{for } 1 \leq u \leq 2 \\ T\left(\frac{u}{2}\right) + 1, & \text{for } 2 < u \leq 3 \end{cases}$$

satisfies the hypothesis of the Lemma 22.1 (for any $p \in \boldsymbol{R}$) with $I = (2, 3]$, $g: I \to \{1\}$, $G = g$, and $S = \{r\}$ where $r: I \to \boldsymbol{R}$ is defined by $r(u) = u/2$ for all $u \in I$, so $I^* = \emptyset$.

Application of the Lemma. We use Lemma 22.1 only in the proof of Lemma 26.1, where R is an admissible recurrence satisfying the modified Leighton Hypothesis relative to some $\varepsilon > 0$. The incremental cost has a tame extension by definition of an admissible recurrence. Lemma 20.9 implies R satisfies the strong ratio condition, so R is proper, i.e., is a divide-and-conquer recurrence. The inequalities of Lemma 22.1 are applied with p as the Akra-Bazzi exponent and only for certain $x \in I$ that are guaranteed to satisfy $r(x) \in I$ for all $r \in S$, so $r(x) \in domain(G)$ for all such x and r.

23. Partition of the Recursion Set

The claimed proof of Leighton's Theorem 2 uses an indexed partition of the domain, $[1, \infty)$, of certain recurrences into non-empty, disjoint, bounded subintervals. (The initial subinterval is closed; the others are left-open, right-closed, unit intervals.) The argument proceeds by strong induction on the index and relies on an asserted property of the partition. As explained in Section 19, the hypothesis of Theorem 2 is insufficient to guarantee that the partition has the required property.

Our proof in Section 26 of Lemma 26.1 is an adaptation of Leighton's argument with an analogous partition. Lemma 23.2 implies the existence of a partition with the necessary properties, including synergy with Lemma 22.1, which is also used in the proof of Lemma 26.1.

We partition the recursion set rather than the recurrence's domain because the strong Akra-Bazzi condition is more naturally a statement about the behavior of a recurrence's solution on the recursion set than on the recurrence's domain. Unlike Leighton's Theorem 2, the hypothesis of Lemmas 26.1 does not require the recursion set to be an interval. Furthermore, the elements of our partition are not necessarily intervals.

Lemmas 26.1 is applicable to admissible recurrences that satisfy the modified Leighton hypothesis. Lemma 20.9 says all such recurrences also satisfy the strong ratio condition. In particular they must be proper, i.e., they are divide-and-conquer recurrences.

Lemma 23.1. Suppose R is a divide-and-conquer recurrence that satisfies the strong ratio condition. Let I be the recursion set of R and let $x_0 = \inf I$. Then there exists a real number $z > x_0$ such that each dependency of R maps

$$I \cap (z + j - 1, z + j]$$

into

$$I \cap [x_0, z + j - 1]$$

for each positive integer j.

Proof. By definition of a semi-divide-and-conquer recurrence, the set I is non-empty and has a positive lower bound. Therefore, x_0 is a positive real number.

Satisfaction of the ratio condition means there exist real numbers α and β such that

$$0 < \alpha \leq \beta < 1$$

and

$$\alpha x \leq r(x) \leq \beta x$$

for all $x \in I$ and each dependency r. Let

$$z = \max\left(\frac{x_0}{\alpha}, \frac{\beta}{1-\beta}\right),$$

so z is a real number such that

$$z \geq \frac{x_0}{\alpha} > x_0$$

and

$$z \geq \frac{\beta}{1-\beta} > 0.$$

Let j be a positive integer, so $z + j - 1 \geq z$. The function

$$t \mapsto \frac{t}{t+1}$$

on $\boldsymbol{R}^+$ is increasing, so

$$\frac{z+j-1}{z+j} \geq \frac{z}{z+1} \geq \frac{\frac{\beta}{1-\beta}}{\frac{\beta}{1-\beta}+1} = \beta.$$

Let D be the domain of the recurrence R. If r is a dependency of R and

$$x \in I \cap (z+j-1, z+j],$$

then $r(x) \in D$ and

$$x_0 \leq \alpha z < \alpha x \leq r(x) \leq \beta x \leq \left(\frac{z+j-1}{z+j}\right)(z+j) = z+j-1,$$

which implies

$$r(x) \in D \cap (x_0, z+j-1].$$

The recursion set I is an upper subset of D by definition of a semi-divide-and-conquer recurrence, so $D \cap (x_0, \infty) \subseteq I$. Therefore,

$$r(x) \in I \cap (x_0, z+j-1] \subseteq I \cap [x_0, z+j-1]$$

as claimed. □

The following minor variation on Lemma 23.1 is slightly more convenient for proving Lemma 26.1:

Lemma 23.2. Suppose R is a divide-and-conquer recurrence that satisfies the strong ratio condition. Let I be the recursion set of R. Then there exists a non-empty lower subset S of $\boldsymbol{N}$, a partition Π of I into non-empty, disjoint, bounded subsets, and a bijection $\pi: S \to \Pi$ such that for each dependency r of R and each positive element n of S,

$$r(I_n) \subseteq \bigcup_{j=0}^{n-1} I_j$$

where I_t denotes $\pi(t)$ for all $t \in S$.

Proof. Let $x_0 = \inf I$. By definition of a semi-divide-and-conquer recurrence, the set I is non-empty and has a positive lower bound. Therefore, x_0 is a positive real number.

Let z be as in Lemma 23. In particular, $z > x_0$. Define

$$A_0 = I \cap [x_0, z],$$

so A_0 is non-empty. Let

$$A_m = I \cap (z + m - 1, z + m]$$

for each positive integer m. The sets $A_0, A_1, A_2, \ldots$ are disjoint and bounded. Furthermore,

$$I = \bigcup_{j=0}^{\infty} A_j.$$

Lemma 23.1 says each dependency of R maps A_m into $I \cap [x_0, z + m - 1]$, i.e.,

$$\bigcup_{j=0}^{m-1} A_j,$$

for each positive integer m. Let

$$W = \{j \in \boldsymbol{N} : A_j \neq \emptyset\}$$

and

$$\Pi = \{A_j : j \in W\},$$

so Π is a partition of I into non-empty, disjoint, bounded subsets. Observe that $0 \in W$ since $A_0 \neq \emptyset$. In particular, W is non-empty.

The set W is a subset of the countable set $\boldsymbol{N}$ and is therefore countable. Either $|W| = |\boldsymbol{N}|$ or W is finite. There exists an order preserving bijection $\lambda: S \to W$ for some non-empty lower subset S of $\boldsymbol{N}$. Either $S = \boldsymbol{N}$ or S is finite. Define a bijection $\pi: S \to \Pi$ by $\pi(t) = A_{\lambda(t)}$, i.e., $I_t = A_{\lambda(t)}$ for all $t \in S$. Non-emptiness of S implies $0 \in S$, i.e.,

$$\min S = 0 = \min W.$$

Then $\lambda(0) = 0$, i.e., $I_0 = A_0$ because $\lambda: S \to W$ is an order preserving bijection. Suppose n is a positive element of S, so

$$\lambda(n) > \lambda(0) = 0.$$

Define

$$Y = \{i \in W : i < \lambda(n)\}.$$

Then

$$Y = \lambda(\{j \in S : j < n\}) = \lambda(\{j \in \boldsymbol{N} : j < n\})$$

because $\lambda: S \to W$ is an order preserving bijection and S is a lower subset of $\boldsymbol{N}$ containing n. Therefore,

$$r(I_n) = r(A_{\lambda(n)}) \subseteq \bigcup_{i=0}^{\lambda(n)-1} A_i = \bigcup_{i \in Y} A_i = \bigcup_{j=0}^{n-1} A_{\lambda(j)} = \bigcup_{j=0}^{n-1} I_j$$

for each dependency r of R. □

Initial element of the partition. Let R be a divide-and-conquer recurrence that satisfies the strong ratio condition. Also suppose the incremental cost of R has a tame extension G. Let I_0 be as in Lemma 23.2, so I_0 is a bounded subset of I. The unique solution of R and the Akra-Bazzi estimate for R relative to G are locally $\Theta(1)$. (See Lemmas 2.2(2), 9.6, and 20.2.) Therefore, they are $\Theta(1)$ on I_0 as required by the base case of an inductive argument analogous to the argument for Theorem 2 in [Le]. (See the proof of Lemma 26.1.)

Induction on the index of a partition element. Our partition of the recursion set may be finite, so induction on the index set S (which is a non-empty lower subset of $\boldsymbol{N}$) requires a slight amount of care. The relevant strong induction principle is as follows: Let L^* be a subset of a lower subset L of $\boldsymbol{N}$. If $0 \in L^*$ (which implies $0 \in L$, i.e., L is non-empty) and $n + 1 \in L^*$ for all $n \in \boldsymbol{N}$ that satisfy $n + 1 \in L$ and

$$\boldsymbol{N} \cap [0, n] \subseteq L^*,$$

then $L^* = L$. We prove this principle by standard strong induction: Inclusion of 0 in L^* implies

$$0 \in L^* \cup (\boldsymbol{N} \backslash L).$$

Suppose $m \in \boldsymbol{N}$ such that

$$\boldsymbol{N} \cap [0, m] \subseteq L^* \cup (\boldsymbol{N} \backslash L).$$

If $m + 1 \notin L$, then

$$m + 1 \in \boldsymbol{N} \backslash L \subseteq L^* \cup (\boldsymbol{N} \backslash L).$$

Now suppose instead that $m + 1 \in L$. Then

$$\boldsymbol{N} \cap [0, m] \subseteq L$$

because L is a lower subset of $\boldsymbol{N}$. Therefore,

$$\boldsymbol{N} \cap [0, m] \subseteq L \cap \big(L^* \cup (\boldsymbol{N} \backslash L)\big) = L^*,$$

which combines with $m + 1 \in L$ to imply $m + 1 \in L^*$, so

$$m + 1 \in L^* \cup (\boldsymbol{N} \backslash L).$$

Thus

$$m + 1 \in L^* \cup (\boldsymbol{N} \backslash L)$$

regardless of whether $m + 1 \in L$. By (strong) induction,

$$\boldsymbol{N} \subseteq L^* \cup (\boldsymbol{N} \backslash L)$$

(i.e., $\boldsymbol{N} = L^* \cup (\boldsymbol{N} \backslash L)$). We conclude from $L \subseteq \boldsymbol{N}$ that

$$L \subseteq L^* \cup (\boldsymbol{N} \backslash L),$$

which implies $L \subseteq L^*$. Then $L^* = L$ since $L^* \subseteq L$ by hypothesis. The principle is proved.

Alternative formulation. Of course, the inductive principle for lower subsets of $\boldsymbol{N}$ may be restated as follows: If A is a lower subset of $\boldsymbol{N}$, and A^* is a subset of A such that $n \in A^*$ for all $n \in A$ satisfying

$$A \cap [0, n - 1] \subseteq A^*,$$

then $A^* = A$. Observe that

$$A \cap [0, -1] = A \cap \emptyset = \emptyset \subseteq A^*,$$

so the hypothesis implies $0 \in A^*$ if $0 \in A$, i.e., A is non-empty. Of course, A^* is empty if A is empty.

24. False Inequalities in Inductive Steps of Leighton's Theorem 2 When $p < 0$

Leighton's Theorem 2 is applicable to recurrences of the form

$$T(x) = \begin{cases} \Theta(1), & \text{for } 1 \le x \le x_0 \\ \displaystyle\sum_{i=1}^{k} a_i T\big(b_i x + h_i(x)\big) + g(x), & \text{for } x > x_{0.} \end{cases}$$

that satisfy various requirements. In particular, condition 1 of that proposition requires $0 < b_i < 1$ and $x_0 \ge 1/b_i$. As explained in Section 19, those particular properties combine with condition 4(a) of Theorem 2 to imply $b_i \log^{1+\varepsilon} x_0 > 1$ for each index i when $p \ne 0$ (and also when $p = 0$ if we consider 0^0 to be undefined). Here ε is a positive real number satisfying conditions 2 and 4 of Theorem 2. As usual, p is the unique real number satisfying

$$\sum_{i=1}^{k} a_i b_i^p = 1.$$

The real-valued functions g and $h_1, \ldots, h_k$ have domains containing $[1, \infty)$. Condition 2 says $|h_i(x)| \le x/\log^{1+\varepsilon} x$ for each index i and all $x \ge x_0$. If $p < 0$, then the function $t \to t^p$ on $\boldsymbol{R}^+$ is strictly decreasing and

$$b_i \log^{1+\varepsilon} x \ge b_i \log^{1+\varepsilon} x_0 > 1$$

for all $x \ge x_0$, so

$$0 < 1 - \frac{1}{b_i \log^{1+\varepsilon} x} \le 1 + \frac{h_i(x)}{b_i x} \le 1 + \frac{1}{b_i \log^{1+\varepsilon} x}$$

and

$$\left(1 + \frac{1}{b_i \log^{1+\varepsilon} x}\right)^p \le \left(1 + \frac{h_i(x)}{b_i x}\right)^p \le \left(1 - \frac{1}{b_i \log^{1+\varepsilon} x}\right)^p.$$

for all such x. The inductive step in Leighton's (incorrect) argument for the existence of $c_5 \in \boldsymbol{R}^+$ such that

$$T(x) \geq c_5 x^p \left(1 + \frac{1}{\log^{\varepsilon/2} x}\right)\left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du\right)$$

for all $x > x_0$ implicitly claims that

$$\left(1 + \frac{h_i(x)}{b_i x}\right)^p \geq \left(1 - \frac{1}{b_i \log^{1+\varepsilon} x}\right)^p$$

for all such x. When $p < 0$ and $x > x_0$ the inequality above is equivalent to

$$\left(1 + \frac{h_i(x)}{b_i x}\right)^p = \left(1 - \frac{1}{b_i \log^{1+\varepsilon} x}\right)^p,$$

i.e.,

$$h_i(x) = -\frac{x}{\log^{1+\varepsilon} x}.$$

The inductive step in Leighton's (incorrect) argument for the existence of $c_6 \in \boldsymbol{R}^+$ such that

$$T(x) \leq c_6 x^p \left(1 - \frac{1}{\log^{\varepsilon/2} x}\right)\left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du\right)$$

for all $x > x_0$ implicitly claims that

$$\left(1 + \frac{h_i(x)}{b_i x}\right)^p \leq \left(1 + \frac{1}{b_i \log^{1+\varepsilon} x}\right)^p$$

for all $x > x_0$. When $p < 0$ and $x > x_0$, the inequality above is equivalent to

$$\left(1 + \frac{h_i(x)}{b_i x}\right)^p = \left(1 + \frac{1}{b_i \log^{1+\varepsilon} x}\right)^p,$$

i.e.,

$$h_i(x) = \frac{x}{\log^{1+\varepsilon} x}.$$

However,

$$\frac{x}{\log^{1+\varepsilon} x} \neq -\frac{x}{\log^{1+\varepsilon} x}$$

for all such x. Therefore, at least one of the inductive steps is incorrect when $p < 0$.

Of course, there exist recurrences that satisfy the hypothesis of Leighton's Theorem 2 but have negative values of p. An example is provided in Sections 13 with $p = -1$; in particular, p is negative.

25. Adjustment of Inequalities for Sign of p

Section 20 defines the technical condition, which plays the same role for Lemma 26.1 (and indirectly for Lemma 20.8) as Leighton's condition 4 (and part of his condition 1) does for Theorem 2 of [Le]. However, the technical condition uses $|p|$ where conditions 4(a) and 4(b) of Leighton's Theorem 2 uses p, the Akra-Bazzi exponent. In conjunction with Lemma 25.3, the change enables a resolution of issues raised in the preceding section about negative Akra-Bazzi exponents.

We note that Theorem 2 of [Le] cannot be remedied by merely replacing p with $|p|$ in conditions 4(a) and 4(b). For example, Section 15 describes a finitely recursive counterexample to Theorem 2 with $p = 0$, i.e., $|p| = p$. With a slight modification to the recurrence of Section 13 when $x_0 = 10000$, we can also produce an infinitely recursive counterexample with $p = 0$. It suffices to let $a = 1$ and

$$\varepsilon = \frac{\log 100}{\log\log 10000} - 1 \approx 1.074$$

as in Section 15. Satisfaction of the hypothesis of Theorem 2 follows from the same arguments as in Section 15. With a few obvious changes, the analysis in Section 13 remains valid. In particular, there exists a solution T of the recurrence that is unbounded on every open set in (x_0, ∞). The Akra-Bazzi formula for T reduces to

$$T(x) = \Theta\left(1 + \int_1^x \frac{du}{u}\right) = \Theta(\log x),$$

which is false.

Lemma 25.1. If

$$(x_0, b_1, \dots, b_k, p, \varepsilon)$$

satisfies the technical condition, then $b_i x_0 > e$ for all $i \in \{1, \dots, k\}$.

Proof. Let $i \in \{1, \dots, k\}$. Parts (1) and (3) of the technical condition imply $x_0 > e$ and

$$b_i \log x_0 > 1.$$

Define $f\colon (1,\infty) \to \boldsymbol{R}$ by $f(t) = t/\log t$, so f is differentiable. The derivative

$$f'(t) = \frac{\log t - 1}{\log^2 t}$$

is positive on (e,∞). The mean value theorem implies

$$f(x_0) > f(e) = e,$$

so

$$b_i x_0 = (b_i \log x_0) f(x_0) > e.$$

□

Lemma 25.2. Suppose

$$R = (D, I, a_1, \ldots, a_k, b_1, \ldots, b_k, f, g, h_1, \ldots, h_k)$$

is an admissible recurrence that satisfies the modified Leighton hypothesis relative to some $\varepsilon > 0$. Let p be the Akra-Bazzi exponent of R. Also let $x \in I$ and $i \in \{1, \ldots, k\}$. Define

$$w = \frac{1}{\log^{\varepsilon/2}\left(b_i x + \frac{x}{\log^{1+\varepsilon} x}\right)},$$

$$y = 1 + \frac{h_i(x)}{b_i x},$$

and

$$z = \frac{1}{\log^{\varepsilon/2} x}.$$

Then

$$y > 0.$$

If $p \geq 0$, then

$$y^p(1 + w) > 1 + z$$

and

$$y^p(1 - w) < 1 - z.$$

If $p \leq 0$, then

$$y^p(1 + w)^{-1} < (1 + z)^{-1}$$

and

$$y^p(1 - w)^{-1} > (1 - z)^{-1}.$$

Proof. Let $x_0 = \inf I$, so $x_0 > 0$ by definition of a semi-divide-and-conquer recurrence. Of course, $x \geq x_0$. Satisfaction by R of the modified Leighton hypothesis by relative to ε implies

$$(x_0, b_1, \ldots, b_k, p, \varepsilon)$$

satisfies the technical condition. Recall from the discussion in Section 20 after the definition of the technical condition that all denominators appearing in the definitions of

w, y, and z are defined as positive real numbers; in particular, the denominators are non-zero, so w, y, and z are defined as real numbers. Furthermore, $w > 0$ and $z > 0$. Part (1) of the technical condition says $0 < b_i < 1$. Lemma 25.1 implies $b_i x_0 > e$, so $x > b_i x > e$. Observe that

$$\frac{1}{w} = \log^{\varepsilon/2}\left(b_i x + \frac{x}{\log^{1+\varepsilon} x}\right) > \log^{\varepsilon/2}(b_i x) > \log^{\varepsilon/2} e = 1,$$

which implies $w < 1$, i.e., $1 - w > 0$. Similarly,

$$\frac{1}{z} = \log^{\varepsilon/2} x > \log^{\varepsilon/2}(e) = 1,$$

which implies $z < 1$, i.e., $1 - z > 0$. Satisfaction by R of the modified Leighton hypothesis relative to ε implies satisfaction of Leighton's noise condition on I relative to ε, so

$$|h_i(x)| \le \frac{x}{\log^{1+\varepsilon} x}.$$

As explained in Section 20 after the definition of the technical condition, parts (1), (2), and (3) of the technical condition imply

$$b_i \log^{1+\varepsilon} x > 1.$$

Define

$$L = 1 - \frac{1}{b_i \log^{1+\varepsilon} x}$$

and

$$U = 1 + \frac{1}{b_i \log^{1+\varepsilon} x}.$$

Then

$$0 < L \le y \le U.$$

In particular, $y > 0$ as claimed. Furthermore,

$$0 < L^{|p|} \le y^{|p|} \le U^{|p|}$$

and

$$0 < U^{-|p|} \le y^{-|p|} \le L^{-|p|}.$$

Part (5) of the technical condition says

$$L^{|p|}(1 + w) > 1 + z$$

and

$$U^{|p|}(1 - w) < 1 - z.$$

Suppose $p \ge 0$, so $|p| = p$. Then

$$L^{|p|} \leq y^p \leq U^{|p|}.$$

Positivity of $1+w$ and $1-w$ imply

$$y^p(1+w) \geq L^{|p|}(1+w)$$

and

$$y^p(1-w) \leq U^{|p|}(1-w).$$

Then

$$y^p(1+w) > 1+z$$

and

$$y^p(1-w) < 1-z$$

as claimed.

Now suppose instead that $p \leq 0$, so $p = -|p|$, which implies

$$U^p \leq y^p \leq L^p.$$

Part (5) of the technical condition combines with $p = -|p|$ and positivity of L, U, $1 \pm w$, and $1 \pm z$ to imply

$$L^p(1+w)^{-1} < (1+z)^{-1}$$

and

$$U^p(1-w)^{-1} > (1-z)^{-1}.$$

Positivity of $(1 \pm w)^{-1}$ implies

$$y^p(1+w)^{-1} \leq L^p(1+w)^{-1}$$

and

$$y^p(1-w)^{-1} \geq U^p(1-w)^{-1},$$

so

$$y^p(1+w)^{-1} < (1+z)^{-1}$$

and

$$y^p(1-w)^{-1} > (1-z)^{-1}$$

as claimed □

As explained in Section 24, Leighton's argument in [Le] for his false Theorem 2 implicitly uses the inequalities

$$\left(1 - \frac{1}{b_i \log^{1+\varepsilon} x}\right)^p \leq \left(1 + \frac{h_i(x)}{b_i x}\right)^p \leq \left(1 + \frac{1}{b_i \log^{1+\varepsilon} x}\right)^p$$

for all $x > x_0$ where x_0 and p are defined as in [Le]. However, at least one of the inequalities is violated if $p < 0$. Our proof of the analogous Lemma 26.1 uses inequalities provided by the next proposition instead.

Lemma 25.3. Let $\varepsilon > 0$, and suppose

$$R = (D, I, a_1, \ldots, a_k, b_1, \ldots, b_k, f, g, h_1, \ldots, h_k)$$

is an admissible recurrence that satisfies the modified Leighton hypothesis relative to ε. Let $x_0 = \inf I$ and let p be the Akra-Bazzi exponent of R. Let $r_i, .., r_k: I \to D$ be the dependencies of R defined by

$$r_j(u) = b_j u + h_j(u)$$

for all $u \in I$ and $j \in \{1, \ldots, k\}$. Define $z, L, U: [x_0, \infty) \to \boldsymbol{R}^+$ as follows:

$$z(t) = \frac{1}{\log^{\varepsilon/2} t}.$$

If $p \geq 0$, then

$$L(t) = 1 + z(t)$$

and

$$U(t) = 1 - z(t).$$

If $p < 0$, then

$$L(t) = \big(1 - z(t)\big)^{-1}$$

and

$$U(t) = \big(1 + z(t)\big)^{-1}.$$

The functions L and U satisfy

$$\frac{1}{2} < U(t) < 1 < L(t) < 2$$

for all $t \in [x_0, \infty)$. Furthermore,

$$\big(r_i(x)\big)^p L\big(r_i(x)\big) > b_i^p x^p L(x)$$

and

$$\big(r_i(x)\big)^p U\big(r_i(x)\big) < b_i^p x^p U(x)$$

for all $x \in I$ and $i \in \{1, \ldots, k\}$ that satisfy $r_i(x) \geq x_0$.

Proof. Satisfaction of the modified Leighton hypothesis by R relative to ε implies

$$(x_0, b_1, \ldots, b_k, p, \varepsilon)$$

satisfies the technical condition. Parts (1), (3), and (4) of the technical condition imply $x_0 > e$ and $\log^{\varepsilon/2} x_0 > 2$. Let $t \in [x_0, \infty)$, so $t > e$ and $\log^{\varepsilon/2} t > 2$. In particular, $\log^{\varepsilon/2} t > 0$, so $z(t)$ is a positive real number as claimed. The inequalities

$$0 < z(t) < \frac{1}{2}$$

imply

$$\frac{1}{2} < 1 - z(t) < 1 < 1 + z(t) < 2$$

and

$$\frac{1}{2} < \left(1 + z(t)\right)^{-1} < 1 < \left(1 - z(t)\right)^{-1} < 2.$$

Thus

$$\frac{1}{2} < U(t) < 1 < L(t) < 2$$

as claimed. In particular, $L(t)$ and $U(t)$ are positive real numbers as claimed.

Suppose $x \in I$ and $i \in \{1, \ldots, k\}$ such that $r_i(x) \geq x_0$. Inclusion of x in I implies $x \geq x_0$, so $x > e$. Part (1) of the technical condition says $b_i > 0$, so $b_i x > 0$ and we may define the real number

$$y = 1 + \frac{h_i(x)}{b_i x}.$$

Lemma 25.2 says $y > 0$, so y^p is defined as a positive real number. Observe that $\log x > 0$, so $\log^{1+\varepsilon} x$ is also defined as a positive real number. Define the real number

$$s = b_i x + \frac{x}{\log^{1+\varepsilon} x}.$$

By definition of the modified Leighton hypothesis, R satisfies Leighton's noise condition on I relative to ε. In particular,

$$r_i(x) \leq s.$$

The numbers $r_i(x)$ and s are contained in the domain, $[x_0, \infty)$, of L and U. Furthermore, L is a decreasing function and U is an increasing function. Those facts combine with Lemma 25.2 to imply

$$y^p L\left(r_i(x)\right) \geq y^p L(s) > L(x)$$

and

$$y^p U\left(r_i(x)\right) \leq y^p U(s) < U(x).$$

We conclude from

$$y = \frac{r_i(x)}{b_i x}$$

that

$$\left(r_i(x)\right)^p L\left(r_i(x)\right) > b_i^p x^p L(x)$$

and

$$\left(r_i(x)\right)^p U\left(r_i(x)\right) < b_i^p x^p U(x)$$

as claimed. □

26. Upper and Lower Bounds for Solutions

The proposition below is analogous to the inductive hypothesis of Leighton's (incorrect) proof of his (false) Theorem 2 in [Le]. Our proof is an adaptation of his argument.

Lemma 26.1. Let $\varepsilon > 0$. Suppose

$$R = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k)$$

is an admissible recurrence satisfying the modified Leighton hypothesis relative to ε, and suppose G is a tame extension of g. Then R has a unique solution T, and there exist positive real numbers λ_1 and λ_2 such that

$$\lambda_1 L(x) A(x) < T(x) < \lambda_2 U(x) A(x)$$

for all $x \in I$ where $A: I \to \boldsymbol{R}^+$ is the Akra-Bazzi estimate for R relative to G, and the functions $L, U: [x_0, \infty) \to \boldsymbol{R}^+$ are defined as in Lemma 25.3 with $x_0 = \inf I$.

Proof. (The existence of a tame extension of g is guaranteed by definition of an admissible recurrence.) Lemma 20.9 implies the recurrence satisfies the strong ratio condition and has a unique solution T, which is locally $\Theta(1)$. Define $r_1, \dots, r_k: I \to D$ by

$$r_i(x) = b_i x + h_i(x)$$

for all $x \in I$ and all $i \in \{1, \dots, k\}$. The ratio condition implies R is proper, i.e., $r_i(x) < x$ for each such x and i. By definition of a semi-divide-and-conquer recurrence, $x_0 > 0$ and g is a non-negative function with domain I. Furthermore, G is non-negative by Lemma 10.1(1) (or Lemma 2.7). By definition of a tame function, $domain(G)$ is a non-empty, positive interval. Of course, $x_0 \geq \inf domain(G)$ since $I = domain(g)$ is contained in $domain(G)$. Observe that I is contained in $[x_0, \infty)$, which is the domain of L and U.

Let p be the Akra-Bazzi exponent of R. Corollary 10.3 implies the function

$$u \mapsto G(u)/u^{p+1}$$

on $domain(G)$ is tame (in particular, it is locally Riemann integrable.) Given $\mu_1 \in I \cup \{x_0\}$ and $\mu_2 \in I$ with $\mu_1 \leq \mu_2$, either $[\mu_1, \mu_2] \subseteq domain(G)$ or the following conditions are satisfied:

$$\mu_2 > \mu_1 = x_0 = \inf domain(G) \notin domain(G),$$

$$[x_0, \mu_2] \cap domain(G) = (x_0, \mu_2],$$

and the improper integral

$$\int_{x_0}^{\mu_2} \frac{G(u)}{u^{p+1}} du = \lim_{l \to x_0^+} \int_{l}^{\mu_2} \frac{G(u)}{u^{p+1}} du$$

converges by Lemma 10.5. (Thus any improper integrals in this proof converge.)

By Lemma 23.2, there exists a non-empty lower subset S of $\boldsymbol{N}$, a partition Π of I into non-empty, disjoint, bounded subsets, and a bijection $\pi: S \to \Pi$ such that for all $i \in \{1, \ldots, k\}$ and each positive element m of S, we have

$$r_i(I_m) \subseteq \bigcup_{j=0}^{m-1} I_j \subset I \subseteq domain(G)$$

where I_t denotes $\pi(t)$ for all $t \in S$. By Lemma 22.1, there exist positive real numbers c_1 and c_2 such that

$$c_1 g(y) \leq y^p \int_{r_i(y)}^{y} \frac{G(u)}{u^{p+1}} du \leq c_2 g(y)$$

for all $y \in I \backslash I_0$ and all $i \in \{1, \ldots, k\}$.

The function A is locally $\Theta(1)$ by Lemma 20.2, so there exist positive real numbers c_3 and c_4 such that

$$A(I_0) \subseteq [c_3, c_4].$$

Since T is locally $\Theta(1)$, there exist positive real numbers c_5 and c_6 such that

$$T(I_0) \subseteq [c_5, c_6].$$

Define positive real numbers

$$\lambda_1 = \frac{1}{2} \cdot \min\left\{\frac{c_5}{c_4}, \frac{1}{c_2}\right\}$$

and

$$\lambda_2 = 2 \cdot \max\left\{\frac{c_6}{c_3}, \frac{1}{c_1}\right\}.$$

Let S^* be the set of all $\beta \in S$ that satisfy

$$\lambda_1 L(v) A(v) < T(v) < \lambda_2 U(v) A(v)$$

for all $v \in I_\beta$. Lemma 25.3 implies

$$U(I) \subseteq \left(\frac{1}{2}, 1\right)$$

and

$$L(I) \subseteq (1,2),$$

so

$$\lambda_1 L(w)A(w) < \frac{c_5}{2c_4} \cdot 2 \cdot c_4 = c_5 \leq T(w) \leq c_6 = \frac{2c_6}{c_3} \cdot \frac{1}{2} \cdot c_3 < \lambda_2 U(w)A(w)$$

for all $w \in I_0$. Therefore, $0 \in S^*$. Suppose $n \in \boldsymbol{N}$ such that $n + 1 \in S$ and

$$\boldsymbol{N} \cap [0, n] \subseteq S^*.$$

(There is no such n if $S = \{0\}$, i.e., $I = I_0$). Let $z \in I_{n+1}$, so

$$r_i(z) \in \bigcup_{j=0}^{n} I_j \subset I \subseteq [x_0, \infty)$$

for all $i \in \{1, \ldots, k\}$. Then

$$\lambda_1 L\big(r_i(z)\big)A\big(r_i(z)\big) < T\big(r_i(z)\big) < \lambda_2 U\big(r_i(z)\big)A\big(r_i(z)\big)$$

for each such i. Since T is a solution of R,

$$T(z) \; = \sum_{i=1}^{k} a_i T\big(r_i(z)\big) + g(z).$$

Positivity of $a_1, \ldots, a_k$ implies

$$T(z) \; > \sum_{i=1}^{k} a_i \lambda_1 L\big(r_i(z)\big)A\big(r_i(z)\big) + g(z),$$

i.e.,

$$T(z) \; > \sum_{i=1}^{k} a_i \lambda_1 L\big(r_i(z)\big)\big(r_i(z)\big)^p \left(1 + \int_{x_0}^{r_i(z)} \frac{G(u)}{u^{p+1}} du\right) + g(z).$$

Lemma 25.3 implies

$$L\big(r_i(z)\big)\big(r_i(z)\big)^p > b_i^p z^p L(z)$$

for each index i. Then positivity of $a_1, \ldots, a_k$, and λ_1 combines with non-negativity of the integrand to imply

$$T(z) \; > \sum_{i=1}^{k} a_i \lambda_1 b_i^p z^p L(z) \left(1 + \int_{x_0}^{r_i(z)} \frac{G(u)}{u^{p+1}} du\right) + g(z),$$

i.e.,

$$T(z) > \sum_{i=1}^{k} a_i b_i^p \lambda_1 L(z) z^p \left(1 + \int_{x_0}^{z} \frac{G(u)}{u^{p+1}} du - \int_{r_i(z)}^{z} \frac{G(u)}{u^{p+1}} du\right) + g(z).$$

The defining inequality for c_2 combines with positivity of $a_1, \ldots, a_k, b_1, \ldots, b_k, \lambda_1, L(z)$, and z to imply

$$T(z) > \sum_{i=1}^{k} a_i {b_i}^p \lambda_1 L(z) z^p \left(1 + \int_{x_0}^{z} \frac{G(u)}{u^{p+1}} du - \frac{c_2 g(z)}{z^p}\right) + g(z).$$

Recall that

$$\sum_{i=1}^{k} a_i {b_i}^p = 1$$

by definition of the Akra-Bazzi exponent, p, of R. Therefore,

$$T(z) > \lambda_1 L(z) z^p \left(1 + \int_{x_0}^{z} \frac{G(u)}{u^{p+1}} du - \frac{c_2 g(z)}{z^p}\right) + g(z)$$

$$= \lambda_1 L(z)\big(A(z) - c_2 g(z)\big) + g(z)$$

$$= \lambda_1 L(z) A(z) + \big(1 - \lambda_1 c_2 L(z)\big) g(z).$$

Recall that $0 < \lambda_1 \leq 1/(2c_2)$ and $L(z) < 2$, so $\lambda_1 c_2 L(z) < 1$. Non-negativity of g implies

$$\big(1 - \lambda_1 c_2 L(z)\big) g(z) \geq 0.$$

Therefore,

$$T(z) > \lambda_1 L(z) A(z).$$

We now establish an upper bound for $T(z)$ in the same fashion. Positivity of $a_1, \ldots, a_k$ combines with

$$T\big(r_i(z)\big) < \lambda_2 U\big(r_i(z)\big) A\big(r_i(z)\big)$$

for all $i \in \{1, \ldots, k\}$ and

$$T(z) = \sum_{i=1}^{k} a_i T\big(r_i(z)\big) + g(z)$$

to imply

$$T(z) < \sum_{i=1}^{k} a_i \lambda_2 U\big(r_i(z)\big) A\big(r_i(z)\big) + g(z),$$

i.e.,

$$T(z) < \sum_{i=1}^{k} a_i \lambda_2 U\big(r_i(z)\big) \big(r_i(z)\big)^p \left(1 + \int_{x_0}^{r_i(z)} \frac{G(u)}{u^{p+1}} du\right) + g(z).$$

Lemma 25.3 implies

$$U\big(r_i(z)\big)\big(r_i(z)\big)^p < b_i^p z^p U(z)$$

for each index i. Then positivity of $a_1, \dots, a_k$, and λ_2 combines with non-negativity of the integrand to imply

$$T(z) < \sum_{i=1}^{k} a_i \lambda_2 b_i^p z^p U(z) \left(1 + \int_{x_0}^{r_i(z)} \frac{G(u)}{u^{p+1}} du\right) + g(z),$$

i.e.,

$$T(z) < \sum_{i=1}^{k} a_i b_i^p \lambda_2 U(z) z^p \left(1 + \int_{x_0}^{z} \frac{G(u)}{u^{p+1}} du - \int_{r_i(z)}^{z} \frac{G(u)}{u^{p+1}} du\right) + g(z).$$

The defining inequality for c_1 combines with positivity of $a_1, \dots, a_k, b_1, \dots, b_k, \lambda_2, U(z)$, and z to imply

$$T(z) < \sum_{i=1}^{k} a_i {b_i}^p \lambda_2 U(z) z^p \left(1 + \int_{x_0}^{z} \frac{G(u)}{u^{p+1}} du - \frac{c_1 g(z)}{z^p}\right) + g(z).$$

We conclude from

$$\sum_{i=1}^{k} a_i {b_i}^p = 1$$

that

$$T(z) < \lambda_2 U(z) z^p \left(1 + \int_{x_0}^{z} \frac{G(u)}{u^{p+1}} du - \frac{c_1 g(z)}{z^p}\right) + g(z)$$

$$= \lambda_2 U(z)\big(A(z) - c_1 g(z)\big) + g(z)$$

$$= \lambda_2 U(z) A(z) + \big(1 - \lambda_2 c_1 U(z)\big) g(z).$$

Recall that $\lambda_2 \geq 2/c_1 > 0$ and $U(z) > 1/2$, so $\lambda_2 c_1 U(z) > 1$. Non-negativity of g implies

$$\big(1 - \lambda_2 c_1 U(z)\big) g(z) \leq 0,$$

so

$$T(z) < \lambda_2 U(z) A(z).$$

Therefore, $n + 1 \in S^*$. We conclude that $S^* = S$. Let $x \in I$, so $x \in I_\alpha$ for some $\alpha \in S$. Then $\alpha \in S^*$, so

$$\lambda_1 L(x) A(x) < T(x) < \lambda_2 U(x) A(x)$$

as claimed. □

As promised in Section 20, we now prove Lemma 20.8. For convenience, we repeat the statement:

Lemma 20.8. Let $\varepsilon > 0$. If R is an admissible recurrence that satisfies the modified Leighton hypothesis relative to ε, then R has a unique solution T, which satisfies the strong Akra-Bazzi condition relative to R and g for each tame extension g of the incremental cost of R.

Proof. Let I be the recursion set of R, and define $L, U\colon [\inf I, \infty) \to \boldsymbol{R}^+$ as in Lemma 25.3, which implies

$$U(x) < 1 < L(x)$$

for all $x \in I$. Let g be any tame extension of the incremental cost of R. (There exists a tame extension by definition of an admissible recurrence.) Let $A\colon I \to \boldsymbol{R}^+$ be the Akra-Bazzi estimate for R relative to g. Lemma 26.1 says R has a unique solution T and there exist positive real numbers λ_1 and λ_2 such that

$$\lambda_1 L(x)A(x) < T(x) < \lambda_2 U(x)A(x)$$

for all $x \in I$, so

$$\lambda_1 A(x) < T(x) < \lambda_2 A(x)$$

for all such x. Thus T satisfies the strong Akra-Bazzi condition relative to R and g. □

27. Preliminaries to Lemma 20.7

This section contains results used by the proof of Lemma 20.7. We start with some minor observations.

Lemma 27.1. If $\alpha < r < \beta$ are real numbers, then there exists $u \in (0,1)$ such that

$$1 + \alpha t < (1+t)^r < 1 + \beta t$$

and

$$1 - \beta t < (1-t)^r < 1 - \alpha t$$

for all $t \in (0, u)$.

Proof. Define the differentiable function $f: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ by $f(x) = x^r$, so $f'(1) = r$. Since $\alpha < f'(1) < \beta$, there exists $u \in (0,1)$ such that

$$\alpha < \frac{f(1+t) - f(1)}{t} < \beta$$

and

$$\alpha < \frac{f(1-t) - f(1)}{-t} < \beta$$

for all $t \in (0, u)$, so

$$f(1) + \alpha t < f(1+t) < f(1) + \beta t$$

and

$$f(1) - \beta t < f(1-t) < f(1) - \alpha t$$

for all such t. (Of course, $1 + t, 1 - t \in domain(f)$.) □

***p*, *b*, *ε*, *δ*, and *μ*.** For the remainder of this section, p, b, and ε are real numbers such that

$$0 < b < 1$$

and

$$\varepsilon > 0.$$

In addition,

$$\mu = e^{\left(\left(\frac{1}{1-b}\right)^{\left(\frac{1}{1+\varepsilon}\right)}\right)}$$

and

$$\delta = \max\{\mu, e^{1/b}\}.$$

Observe that $1/(1-b) > 1$, and $1/(1+\varepsilon) > 0$, so

$$\left(\frac{1}{1-b}\right)^{\left(\frac{1}{1+\varepsilon}\right)} > 1,$$

which implies $\mu > e$ and $\log \mu > 1$. Furthermore,

$$\log^{1+\varepsilon} \mu = \frac{1}{1-b}.$$

Lemma 27.2. $\delta > e/b$.

Proof. The differentiable function $f: \boldsymbol{R} \to \boldsymbol{R}$ defined by

$$f(t) = e^t - et$$

satisfies $f(1) = 0$, while its derivative, $t \mapsto e^t - e$, is positive on $(1, \infty)$. Therefore, $f(t) > 0$ for all $t > 1$. In particular,

$$e^{1/b} > \frac{e}{b}.$$

The lemma follows from

$$\delta \geq e^{1/b}.$$

□

***A*, *B*, and *C*.** In this section, A, B, and C are real-valued functions on (δ, ∞) defined by

$$A(x) = \frac{1}{\log^{\varepsilon/2}\left(bx + \frac{x}{\log^{1+\varepsilon} x}\right)},$$

$$B(x) = \frac{1}{\log^{\varepsilon/2} x},$$

and

$$C(x) = \frac{1}{b \log^{1+\varepsilon} x}.$$

Lemma 27.3. If $x > \delta$, then

$$0 < B(x) < A(x) < 1$$

and

$$0 < C(x) < 1.$$

Proof. By definition, $\delta \geq e^{1/b}$ and $b \in (0,1)$, so $x > e^{1/b} > e$ and

$$\log x > 1/b > 1.$$

Positivity of ε implies each element α of $\{\varepsilon, \varepsilon/2, 1+\varepsilon\}$ is positive and satisfies $\log^{\alpha} x > 1$. In particular, $B(x) > 0$. Positivity of b combines with $\log^{\varepsilon} x > 1$ and $\log x > 1/b$ to imply

$$b \log^{1+\varepsilon} x > b \log x > 1,$$

so $0 < C(x) < 1$. Let

$$t = bx + \frac{x}{\log^{1+\varepsilon} x}.$$

Positivity of x and $\log^{1+\varepsilon} x$ implies $t > bx$. Lemma 27.2 combines with $x > \delta$ and $b > 0$ to imply $bx > b\delta > e$. Then $t > e$ and $\log t > 1$. Positivity of $\varepsilon/2$ implies $\log^{\varepsilon/2} t > 1$, so $A(x) < 1$.

By definition, $x > \delta \geq \mu > e$, so $\log x > \log \mu > 0$. Positivity of $1+\varepsilon$ and $1-b$ combines with the definition of μ to imply

$$\log^{1+\varepsilon} x > \log^{1+\varepsilon} \mu = \frac{1}{(1-b)} > 0.$$

Now

$$b + \frac{1}{\log^{1+\varepsilon} x} < 1,$$

which combines with positivity of x to imply $t < x$. Then

$$0 < \log^{\varepsilon/2} t < \log^{\varepsilon/2} x$$

follows from $\log x > \log t > 0$ and $\varepsilon > 0$. Therefore, $B(x) < A(x)$. □

We observe from the proof Lemma 27.3 that the definitions of A, B, and C as real-valued functions are valid. The logarithms that appear in the definitions are positive real numbers, so the required powers of the logarithms represent positive real numbers. All denominators that appear in the definitions are positive and therefore non-zero.

Lemma 27.4. If $x > \delta$, then

$$\frac{1 + A(x)}{1 + B(x)} = 1 + \big(A(x) - B(x)\big)\big(1 - B(x)\big) \sum_{n=0}^{\infty} \big(B(x)\big)^{2n}.$$

Proof. Lemma 27.3 implies $0 < B(x) < 1$, so

$$\frac{1 + A(x)}{1 + B(x)} = \big(1 + A(x)\big) \sum_{n=0}^{\infty} \big(-B(x)\big)^{n}$$

$$= 1 + \sum_{n=1}^{\infty} \left(-B(x)\right)^n + A(x) \sum_{n=0}^{\infty} \left(-B(x)\right)^n$$

$$= 1 + \sum_{n=0}^{\infty} \left(-\left(B(x)\right)^{2n+1} + \left(B(x)\right)^{2n+2}\right) + A(x) \sum_{n=0}^{\infty} \left(\left(B(x)\right)^{2n} - \left(B(x)\right)^{2n+1}\right)$$

$$= 1 + \left(\left(B(x)\right)^2 - B(x) + A(x) - A(x)B(x)\right) \sum_{n=0}^{\infty} \left(B(x)\right)^{2n}$$

$$= 1 + \left(A(x) - B(x)\right)\left(1 - B(x)\right) \sum_{n=0}^{\infty} \left(B(x)\right)^{2n}.$$

□

Lemma 27.5.

$$\lim_{x\to\infty} \frac{A(x) - B(x)}{C(x)} = \infty.$$

Proof. Recall that $\delta \geq \mu > e$. For all $x > \delta$, we have $x > \mu > e$, i.e.,

$$\log x > \log \mu > 1,$$

which combines with $\varepsilon > 0$, $b \in (0,1)$, and the definition of μ to imply

$$\log^{1+\varepsilon} x > \log^{1+\varepsilon} \mu = \frac{1}{(1-b)} > 1.$$

For each such x,

$$b < b + \frac{1}{\log^{1+\varepsilon} x} < 1$$

and

$$\log b < \log\left(b + \frac{1}{\log^{1+\varepsilon} x}\right) < 0.$$

Define $\lambda\colon (\delta, \infty) \to \boldsymbol{R}$ by

$$\lambda(x) = \left|\log\left(b + \frac{1}{\log^{1+\varepsilon} x}\right)\right|$$

for all $x > \delta$. Then

$$|\log b| > \lambda(x) > 0$$

and

$$A(x) = \frac{1}{(\log x - \lambda(x))^{\varepsilon/2}}$$

for each such x, so

$$\frac{A(x)-B(x)}{C(x)} = (b\ \log^{1+\varepsilon} x)\left(\frac{1}{(\log x - \lambda(x))^{\varepsilon/2}} - \frac{1}{\log^{\varepsilon/2} x}\right)$$

$$= (b\ \log^{1+\varepsilon} x)\left(\frac{\log^{\varepsilon/2} x - (\log x - \lambda(x))^{\varepsilon/2}}{(\log^{\varepsilon/2} x)(\log x - \lambda(x))^{\varepsilon/2}}\right).$$

For all $x > \delta$, Lemma 27.2 and $b \in (0,1)$ imply

$$\log x > \log \frac{e}{b} = 1 - \log b = 1 + |\log b| > 1 + \lambda(x),$$

and hence

$$\log x > \log x - \lambda(x) > 0.$$

Now positivity of b, ε, and $\log x$ imply

$$\frac{A(x)-B(x)}{C(x)} > (b\ \log^{1+\varepsilon} x)\left(\frac{\log^{\varepsilon/2} x - (\log x - \lambda(x))^{\varepsilon/2}}{\log^{\varepsilon} x}\right)$$

$$= (b \log x)\left(\log^{\varepsilon/2} x - \left(\log x - \lambda(x)\right)^{\varepsilon/2}\right)$$

$$= \left(b \log^{1+\frac{\varepsilon}{2}} x\right)\left(1 - \left(1 - \frac{\lambda(x)}{\log x}\right)^{\varepsilon/2}\right)$$

for each such x. Positivity of ε implies

$$\lim_{t\to\infty} \lambda(t) = |\log b|,$$

so

$$\lim_{t\to\infty} \frac{\lambda(t)}{\log t} = 0.$$

Then Lemma 27.1 implies there exists $y \geq \delta$ such that

$$\left(1 - \frac{\lambda(x)}{\log x}\right)^{\varepsilon/2} < 1 - \frac{\varepsilon}{3}\frac{\lambda(x)}{\log x}$$

for all $x > y$. Positivity of b and $\log x$ implies

$$\frac{A(x)-B(x)}{C(x)} > \left(b \log^{1+\frac{\varepsilon}{2}} x\right)\frac{\varepsilon}{3}\frac{\lambda(x)}{\log x} = \frac{b\varepsilon\, \lambda(x) \log^{\varepsilon/2} x}{3}$$

for each such x. We conclude from

$$\lim_{x\to\infty} \frac{b\varepsilon\lambda(x)}{3} = \frac{b\varepsilon|\log b|}{3} > 0$$

and (recalling that $\log x > 1$ and $\varepsilon > 0$)

$$\lim_{x\to\infty} \log^{\varepsilon/2} x = \infty$$

that

$$\lim_{x\to\infty} \frac{A(x) - B(x)}{C(x)} = \infty.$$

□

For convenience, we include the following simple observation:

Lemma 27.6.

$$\lim_{x\to\infty} \frac{\left(B(x)\right)^2}{C(x)} = \lim_{x\to\infty} \frac{\left(A(x) - B(x)\right)\left(1 - B(x)\right)}{C(x)} = \infty.$$

Proof. Observe that

$$\frac{\left(B(x)\right)^2}{C(x)} = \frac{b \log^{1+\varepsilon} x}{\log^{\varepsilon} x} = b \log x,$$

so positivity of b implies

$$\lim_{x\to\infty} \frac{\left(B(x)\right)^2}{C(x)} = \infty.$$

Positivity of ε implies

$$\lim_{x\to\infty} B(x) = 0,$$

which combines with 27.5 to imply

$$\lim_{x\to\infty} \frac{\left(A(x) - B(x)\right)\left(1 - B(x)\right)}{C(x)} = \infty.$$

□

The next lemma corresponds to part 5(a) of the technical condition for a single index i.

Lemma 27.7. There exists $v \geq \delta$ such that

$$\left(1 - C(x)\right)^{|p|}\left(1 + A(x)\right) > 1 + B(x)$$

for all $x > v$.

Proof. Let $q > |p|$, so $q > 0$. Lemma 27.1 implies there exists $u > 0$ such that

$$(1-t)^{|p|} > 1 - qt$$

for all $t \in (0, u)$. The functions A, B, and C are positive by Lemma 27.3. Observe that

$$\lim_{x\to\infty} C(x) = 0.$$

There exists $y \geq \delta$ such that $C(x) \in (0, u)$ for all $x > y$, so

$$\big(1 - C(x)\big)^{|p|} > 1 - qC(x)$$

for all such x. Since $q > 0$ and hence $1/q > 0$, there exists $z \geq \delta$ such that $C(x) < 1/q$ for all $x > z$, so $0 < qC(x) < 1$ for each such x. Lemma 27.6 and positivity of C imply there exist $s, t \geq \delta$ such that

$$\big(B(x)\big)^2 > qC(x)$$

for all $x > s$ and

$$\big(A(x) - B(x)\big)\big(1 - B(x)\big) > qC(x)$$

for all $x > t$. Define $v = \max\{s, t, y, z\}$, and assume $x > v$. Lemma 27.4 implies

$$\frac{1 + A(x)}{1 + B(x)} = 1 + \big(A(x) - B(x)\big)\big(1 - B(x)\big)\sum_{n=0}^{\infty}\big(B(x)\big)^{2n}.$$

Lemma 27.3 implies $0 < B(x) < 1$, so the infinite series above converges to a positive real number. The inequality $x > z$ implies $0 < qC(x) < 1$, which combines with $x > \max\{s, t\}$ and positivity of the convergent series above to imply

$$\frac{1 + A(x)}{1 + B(x)} > 1 + qC(x)\sum_{n=0}^{\infty}\big(qC(x)\big)^n = \sum_{n=0}^{\infty}\big(qC(x)\big)^n = \frac{1}{1 - qC(x)}.$$

Now the inequality $x > y$ and positivity of $A(x)$, $B(x)$, and $1 - qC(x)$ imply

$$\big(1 - C(x)\big)^{|p|}\big(1 + A(x)\big) > \big(1 - qC(x)\big)\big(1 + A(x)\big) > 1 + B(x),$$

which proves the claim. □

For sake of completeness, we include the following corollary, which corresponds to a strict version of the condition 4(a) of Theorem 2 of [Le] (for a single index i).

Corollary 27.8. There exists $v \geq \delta$ such that

$$\big(1 - C(x)\big)^{p}\big(1 + A(x)\big) > 1 + B(x)$$

for all $x > v$.

Proof. By Lemma 27.7, we may assume $p \neq |p|$, i.e., $p < 0$. Let $v = \delta$ and suppose $x > v$, i.e. $x > \delta$, so x is an element of (δ, ∞), the domain of A, B, and C. Lemma 27.3 implies

$$A(x) > B(x) > 0$$

and

$$0 < C(x) < 1,$$

so

$$\left(1 - C(x)\right)^p > 1$$

and

$$1 + A(x) > 1 + B(x) > 0.$$

Therefore,

$$\left(1 - C(x)\right)^p \left(1 + A(x)\right) > \left(1 - C(x)\right)^p \left(1 + B(x)\right) > 1 + B(x).$$

□

The next lemma corresponds to part 5(b) of the technical condition for a single index i.

Lemma 27.9. There exists $w \geq \delta$ such that

$$\left(1 + C(x)\right)^{|p|} \left(1 - A(x)\right) < 1 - B(x)$$

for all $x > w$.

Proof. Let $q > |p|$, so $q > 0$. Lemma 27.1 implies there exists $u > 0$ such that

$$(1 + t)^{|p|} < 1 + qt$$

for all $t \in (0, u)$. The function C is positive by Lemma 27.3. Observe that

$$\lim_{x \to \infty} C(x) = 0.$$

There exists $y \geq \delta$ such that $C(x) \in (0, u)$ for all $x > y$, so

$$\left(1 + C(x)\right)^{|p|} < 1 + qC(x)$$

for all such x. Lemma 27.5 and positivity of the function C imply there exists $z \geq \delta$ such that

$$A(x) > B(x) + qC(x)$$

for all $x > z$. Let $w = max(y, z)$ and assume $x > w$. Lemma 27.3 says

$$0 < B(x) < A(x) < 1,$$

which combines with $x > z$ to imply

$$\frac{1 + qC(x)}{1 - B(x)} = 1 + \big(B(x) + qC(x)\big)\sum_{n=0}^{\infty}\big(B(x)\big)^n < 1 + A(x)\sum_{n=0}^{\infty}\big(A(x)\big)^n$$

$$= \frac{1}{1 - A(x)}$$

and

$$\big(1 + qC(x)\big)\big(1 - A(x)\big) < 1 - B(x).$$

We conclude from $x > y$ and positivity $1 - A(x)$ that

$$\big(1 + C(x)\big)^{|p|}\big(1 - A(x)\big) < \big(1 + qC(x)\big)\big(1 - A(x)\big) < 1 - B(x).$$

□

For sake of completeness, we include the following corollary, which corresponds to a strict version of condition 4(b) of Theorem 2 of [Le] (for a single index i).

Corollary 27.10. There exists $w \geq \delta$ such that

$$\big(1 + C(x)\big)^p\big(1 - A(x)\big) < 1 - B(x)$$

for all $x > w$.

Proof. By Lemma 27.9, we may assume $p \neq |p|$, i.e., $p < 0$. Let $w = \delta$. If $x > w$, i.e., $x > \delta$, then Lemma 27.3 implies

$$B(x) < A(x) < 1$$

and $C(x) > 0$, so

$$0 < \big(1 + C(x)\big)^p < 1,$$

$$0 < 1 - A(x) < 1 - B(x),$$

and

$$\big(1 + C(x)\big)^p\big(1 - A(x)\big) < \big(1 + C(x)\big)^p\big(1 - B(x)\big) < 1 - B(x).$$

□

28. Proof of Lemma 20.7

As promised in Section 20, we now prove Lemma 20.7. For convenience, we repeat the statement:

Lemma 20.7. If

(1) k is a positive integer,

(2) $b_1, \ldots, b_k$ are real numbers such that $0 < b_i < 1$ for each i,

(3) p is a real number, and

(4) $\varepsilon > 0$,

then there exists a real number x_0 such that $(x_0, b_1, \ldots, b_k, p, \varepsilon)$ satisfies the technical condition.

Proof. Define real numbers $\mu_1, \ldots, \mu_k, \delta_1, \ldots, \delta_k \in (e, \infty)$ by

$$\mu_i = e^{\left(\left(\frac{1}{1-b_i}\right)^{\left(\frac{1}{1+\varepsilon}\right)}\right)}$$

and

$$\delta_i = \max\left(\mu_i, e^{1/b_i}\right).$$

Lemma 27.7 implies there exist real numbers $v_1, \ldots, v_k$ with $v_i \geq \delta_i$ for all $i \in \{1, \ldots, k\}$ such that for each such i the inequality

$$\left(1 - \frac{1}{b_i \log^{1+\varepsilon} x}\right)^{|p|} \left(1 + \frac{1}{\log^{\varepsilon/2}\left(b_i x + \frac{x}{\log^{1+\varepsilon} x}\right)}\right) > 1 + \frac{1}{\log^{\varepsilon/2} x}$$

is satisfied for all $x > v_i$. Lemma 27.9 implies there exist real numbers $w_1, \ldots, w_k$ with $w_i \geq \delta_i$ for all $i \in \{1, \ldots, k\}$ such that for each such i the inequality

$$\left(1+\frac{1}{b_i \log^{1+\varepsilon} x}\right)^{|p|}\left(1-\frac{1}{\log^{\varepsilon/2}\left(b_i x+\frac{x}{\log^{1+\varepsilon} x}\right)}\right)<1-\frac{1}{\log^{\varepsilon/2} x}$$

is satisfied for all $x > w_i$. Define $v = \max\{v_1, \dots, v_k\}$, $w = \max\{w_1, \dots, w_k\}$, and

$$z = e^{(2^{2/\varepsilon})}.$$

Let x_0 be any real number satisfying

$$x_0 > \max\{v, w, z\}.$$

We claim

$$(x_0, b_1, \dots, b_k, p, \varepsilon)$$

satisfies the technical condition. The tuple above is a $(k+3)$-tuple with k a positive integer as required. Conditions (2) and (4) of the proposition are parts (1) and (2), respectively, of the technical condition. Observe that

$$v \geq v_i \geq \delta_i \geq e^{1/b_i}$$

for all $i \in \{1, \dots, k\}$, so $x_0 > e^{1/b_i}$ for each such i, which is part (3) of the technical condition. Also observe that

$$\log^{\varepsilon/2} x_0 > \log^{\varepsilon/2} z = 2,$$

so part (4) of the technical condition is satisfied. The inequalities $x_0 > v$ and $x_0 > w$ imply $x > v$ and $x > w$ for all $x \geq x_0$, so $x > v_i$ and $x > w_i$ for each such x and all $i \in \{1, \dots, k\}$. Therefore, the final part (5) of the technical condition is satisfied. □

29. Solution Insensitivity to Base Case and Incremental Cost

Suppose T is a solution of a divide-and-conquer recurrence

$$R = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f_R, g_R, h_1, \dots, h_k)$$

with base case f_R, incremental cost g_R, and unbounded recursion set I. Let

$$Q = \left(D, I, a_1, \dots, a_k, b_1, \dots, b_k, f_Q, g_Q, h_1, \dots, h_k\right)$$

where f_Q (like f_R) is a $\Theta(1)$ real-valued functions on $D\backslash I$, and g_Q (like g_R) is a non-negative real-valued function on I, so Q is a divide-and-conquer recurrence with base case f_Q and incremental cost g_Q. The recurrences R and Q are identical apart from their base cases and incremental costs, which may be different.

Suppose $g_Q = \Theta(g_R)$ and S is a solution of Q. It is tempting to conclude that $S = \Theta(T)$ as claimed in the examples on page 2 of [Le]. Lemma 29.1 and Corollary 29.2 will justify the conclusion under certain mild conditions. Those propositions also show that other asymptotic relationships between incremental costs are often inherited by solutions. Lemma 29.5 and Corollary 29.6 are analogous results that discard the requirements of a $\Theta(1)$ base case and a non-negative incremental cost.

Section 13 demonstrates that the conclusion is not always justified, even if $f_Q = f_R$ and $g_Q = g_R$. When $x_0 = 10000$, the admissible recurrences defined there are proper, i.e., they are divide-and-conquer recurrences; if the base case is constant with range $\{100\}$, there is a constant solution T with range $\{100\}$. However, the recurrence is infinitely recursive and has other solutions that are unbounded on every open subset of the recursion set. In particular, $S \neq \Theta(T)$ for each such solution S.

We now provide a finitely recursive counterexample:

Example. We will compare two divide-and-conquer recurrences. The first is the admissible recurrence

$$T(x) = \begin{cases} 1, & \text{for } 1 \le x \le 2 \\ T\left(\frac{x}{2}\right) + x, & \text{for } x > 2 \end{cases}$$

with incremental cost $g: (2, \infty) \to \boldsymbol{R}^+$ defined by $g(x) = x$ for all $x \in (2, \infty)$. Let d be the recurrence's depth-of-recursion function. The bounded depth condition is satisfied because

$$d(x) \le \lfloor \log_2 x \rfloor$$

for all $x \in [1, \infty)$ (with equality except when $x \ne 1$ is a power of 2). The recurrence's Akra-Bazzi exponent is zero. Corollary 20.12 implies the recurrence has a unique solution, T, which is positive and satisfies

$$T(x) = \Theta\left(x^0 \left(1 + \int_2^x \frac{u}{u^{o+1}} du\right)\right) = \Theta\left(1 + \int_2^x du\right) = \Theta(\mathrm{x} - 1) = \Theta(x).$$

A simple inductive argument on the depth of recursion shows that $T(x) < 2x$ for all $x \in [1, \infty)$. The second divide-and-conquer recurrence is

$$S(x) = \begin{cases} 1, & \text{for } 1 \le x \le 2 \\ S\left(\frac{x}{2}\right) + x + \frac{1}{x - 2}, & \text{for } x > 2 \end{cases}$$

with incremental cost $f: (2, \infty) \to \boldsymbol{R}^+$ defined by

$$f(x) = x + \frac{1}{x - 2}$$

for all $x \in (2, \infty)$. The two recurrences differ only in their incremental costs, so the second recurrence is also finitely recursive. Corollary 8.5 implies the second recurrence has a unique solution, S, which is positive.

Observe that $f(x) = \Theta(x)$, i.e., $f = \Theta(g)$. We will show that for each pair of real numbers $\alpha > 0$ and $c \ge 1$ (so $[c, \infty)$ is contained in the common domain of S and T) there exists $z > c$ such that $S(z) > \alpha T(z)$. Therefore, $S \ne O(T)$, which implies $S \ne \Theta(T)$.

Define $\beta = \max\{\alpha, 1/8\}$, $m = \max\{c, 2\}$, $n = \lceil \log_2 m \rceil$, and

$$z = 2^n + \frac{1}{8\beta}.$$

Then

$$n \ge \log_2 m \ge \log_2 2 = 1,$$

$$c \le m \le 2^n < z \le 2^n + 1 < 2^{n+1},$$

and

$$S(z) \geq S\left(2+\frac{1}{2^{n+2}\beta}\right) > f\left(2+\frac{1}{2^{n+2}\beta}\right) = 2+\frac{1}{2^{n+2}\beta}+2^{n+2}\beta > 2^{n+2}\beta > 2\beta z.$$

We conclude from $\beta \geq \alpha$ and $z > 0$ that $S(z) > 2\alpha z$. Recall that $T(z) < 2z$, so $S(z) > \alpha T(z)$ as required.

We note that the function f is unbounded on the interval $(2,3)$, so Corollary 2.23 implies f does not have polynomial growth. Lemma 2.2(2) implies f has no polynomial-growth extension, so the recurrence satisfied by S is inadmissible.

Lemma 29.1. Let

$$Q = \left(D, I, a_1, \dots, a_k, b_1, \dots, b_k, f_Q, g_Q, h_1, \dots, h_k\right)$$

and

$$R = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f_R, g_R, h_1, \dots, h_k)$$

be divide-and-conquer recurrences that satisfy the bounded depth condition and are identical apart from their base cases, f_Q and f_R, which may differ, and their incremental costs, g_Q and g_R, which may differ. Assume the recursion set, I, is unbounded. Let S and T be the solutions of Q and R, respectively. Then:

(1) If $g_Q = O(g_R)$ and g_Q is bounded on bounded sets, then there exists a positive real number μ such that $S(x) \leq \mu T(x)$ for all $x \in D$. In particular, $S = O(T)$.

(2) If $g_Q = \Omega(g_R)$ and g_R is bounded on bounded sets, then there exists a positive real number λ such that $\lambda T(x) \leq S(x)$ for all $x \in D$. In particular, $S = \Omega(T)$.

(3) If $g_Q = \Theta(g_R)$, and each of g_Q and g_R is bounded on bounded sets, then there exist positive real numbers λ and μ such that $\lambda T(x) \leq S(x) \leq \mu T(x)$ for all $x \in D$. In particular, $S = \Theta(T)$.

Proof. Corollary 8.5 implies Q and R have unique solutions S and T, respectively, as implicitly claimed. Furthermore, S and T are positive. By definition, g_Q and g_R, are non-negative real-valued functions on I.

By definition, I has a positive lower bound and is a non-empty upper subset of the domain, D, of Q and R. By hypothesis, I is unbounded, so $\sup D = \sup I = \infty$. Therefore, asymptotic notation is defined for S, T, g_Q, and g_R.

We now prove part (1). Suppose $g_Q = O(g_R)$, i.e., there exist a positive real number α and a non-empty upper subset J of I such that $g_Q(r) \leq \alpha g_R(r)$ for all $r \in J$. Also suppose g_Q is bounded on bounded sets. Corollary 9.4 implies S is locally $\Theta(1)$ and each restriction of T to a bounded subset of its domain, D, has a positive lower bound.

Positivity of I implies positivity of $I \setminus J$. Each element of the non-empty set J is a finite upper bound for $I \setminus J$. Therefore, $I \setminus J$ is a bounded subset of D, which implies S is $\Theta(1)$ on $I \setminus J$ and $\inf T(I \setminus J) > 0$. If $I \neq J$, so $I \setminus J$ is non-empty, then $\inf T(I \setminus J)$ and $\sup S(I \setminus J)$ are positive real numbers and we define

$$\beta = \frac{\sup S(I \setminus J)}{\inf T(I \setminus J)};$$

otherwise, define $\beta = 1$. Now β is a positive real number and $S(u) \leq \beta T(u)$ for all $u \in I \setminus J$ (the inequality is vacuously satisfied if $I = J$).

By definition, f_Q and f_R are $\Theta(1)$. If $D \neq I$, so the domain, $D \setminus I$, of f_Q and f_R is non-empty, then $\inf f_R$ and $\sup f_Q$ are positive real numbers, and we define

$$\gamma = \frac{\sup f_Q}{\inf f_R};$$

otherwise, define $\gamma = 1$. Now γ is a positive real number and $f_Q(w) \leq \gamma f_R(w)$ for all $w \in D \setminus I$ (the inequality is vacuously satisfied if $D = I$).

Define

$$\mu = \max\{\alpha, \beta, \gamma\},$$

so μ is a positive real number. Furthermore, $g_Q(r) \leq \mu g_R(r)$ for all $r \in J$ and $S(u) \leq \mu T(u)$ for all $u \in I \setminus J$. We also have

$$S(w) = f_Q(w) \leq \mu f_R(w) = \mu T(w)$$

for all $w \in D \setminus I$. Therefore, $S(z) \leq \mu T(z)$ for all $z \in D \setminus J$.

Let d be the depth-of-recursion function for Q relative to $D \setminus J$. (since Q and R have the same domains and dependencies, d is also the depth-of-recursion function for R relative to $D \setminus J$.) Satisfaction of the bounded depth condition by Q implies Q is finitely recursive relative to $D \setminus I$, which is contained in $D \setminus J$. Then Lemma 8.3 implies Q is also finitely recursive relative to $D \setminus J$, i.e., $d(x) \in \boldsymbol{N}$ for all $x \in D$. Let

$$A = \{n \in \boldsymbol{N} : S(y) \leq \mu T(y) \text{ for all } y \in D \text{ with } d(y) \leq n\}.$$

By definition,

$$D \setminus J = \{t \in D : d(t) = 0\},$$

so $0 \in A$. Let $n \in A$ and suppose $y \in D$ with $d(y) \leq n + 1$. If $d(y) \neq n + 1$, then $d(y) \leq n$, which implies $S(y) \leq \mu T(y)$. If instead $d(y) = n + 1$, then $d(y) > 0$, i.e., $y \in J$. Furthermore,

$$d\big(b_i y + h_i(y)\big) \leq n$$

for all $i \in \{1, \ldots, k\}$. Therefore,

$$S(y) \ = \ \sum_{i=1}^{k} a_i S\big(b_i y + h_i(y)\big) + g_Q(y) \ \le \ \mu \cdot \left(\sum_{i=1}^{k} a_i T\big(b_i y + h_i(y)\big) + g_R(y) \right),$$

i.e.,

$$S(y) \le \mu T(y).$$

Therefore, $n + 1 \in A$. By induction, $A = \boldsymbol{N}$, so $S(x) \le \mu T(x)$ for all $x \in D$. In particular, $S = O(T)$. Part (1) is proved.

We now prove part (2). Suppose $g_Q = \Omega(g_R)$, which implies $g_R = O\big(g_Q\big)$, and assume g_R is bounded on bounded sets. Part (1) implies there exists a real number $\delta > 0$ such that $T(x) \le \delta S(x)$ for all $x \in D$. Define $\lambda = 1/\delta$, so $\lambda > 0$ and $\lambda T(x) \le S(x)$ for all $x \in D$. In particular, $S = \Omega(T)$. Part (2) is proved.

Finally, we prove part (3). Suppose $g_Q = \Theta(g_R)$, so $g_Q = O(g_R)$ and $g_Q = \Omega(g_R)$, and assume each of g_Q and g_R is bounded on bounded sets. Parts (1) and (2) imply there exist positive real numbers λ and μ such that

$$\lambda T(x) \le S(x) \le \mu T(x)$$

for all $x \in D$. In particular, $S = \Theta(T)$. Part (3) is proved. □

Equivalence of bounded depth conditions for $\boldsymbol{Q}$ and $\boldsymbol{R}$. The requirement of Lemma 29.1 that Q and R both satisfy the bounded depth condition is slightly redundant. The two recurrences have the same domain and dependencies, so they have the same depth-of-recursion function. In particular, Q satisfies the bounded depth condition if and only if R satisfies the bounded depth condition.

Incremental costs with polynomial growth. By definition, the recursion set of a divide-and-conquer recurrence has a positive lower bound. If the incremental cost has polynomial growth, then Corollary 2.23 implies the incremental cost is bounded on bounded sets as required by Lemma 29.1.

We now give an interpretation of Lemma 29.1 for integer recurrences:

Corollary 29.2. Let

$$Q = \big(D, I, a_1, \dots, a_k, b_1, \dots, b_k, f_Q, g_Q, h_1, \dots, h_k\big)$$

and

$$R = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f_R, g_R, h_1, \dots, h_k)$$

be divide-and-conquer recurrences that are identical apart from their base cases, f_Q and f_R, which may differ, and their incremental costs, g_Q and g_R, which may differ. Assume the recursion set, I, is unbounded and contains only integers. Let S and T be the solutions of Q and R, respectively. Then:

(1) If $g_Q = O(g_R)$, then there exists a positive real number μ such that $S(n) \leq \mu T(n)$ for all $n \in D$. In particular, $S = O(T)$.

(2) If $g_Q = \Omega(g_R)$, then there exists a positive real number λ such that $\lambda T(n) \leq S(n)$ for all $n \in D$. In particular, $S = \Omega(T)$.

(3) If $g_Q = \Theta(g_R)$, then there exist real positive real numbers λ and μ such that $\lambda T(n) \leq S(n) \leq \mu T(n)$ for all $n \in D$. In particular, $S = \Theta(T)$.

Proof. Lemma 21.1 implies Q and R have unique solutions S and T, respectively, as implicitly claimed. Furthermore, Q and R satisfy the bounded depth condition. Since I is a set of integers, each bounded subset of I is finite and is therefore mapped to finite sets of real numbers by g_Q and g_R. Each such finite set is bounded. The proposition follows from Lemma 29.1. □

Corollary 29.3. Let R be a divide-and-conquer recurrence that satisfies the bounded depth condition and has low noise and an unbounded recursion set. If T is the solution of R, then

$$T(x) = \Omega(x^p)$$

where p is the Akra-Bazzi exponent of R.

Proof. Corollary 8.5 implies R has a unique solution T as implicitly claimed. Let

$$S = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, z, h_1, \dots, h_k)$$

where

$$R = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k)$$

and $z: I \to \{0\}$, i.e., S is the divide-and-conquer recurrence that is identical to R except perhaps for its incremental cost, which is identically zero. In particular, S inherits low noise, an unbounded recursion set, Akra-Bazzi exponent p, and satisfaction of the bounded depth condition from R.

The identically zero function $z^*: (0, \infty) \to \{0\}$ is a tame extension of the incremental cost, z, of S. Existence of a tame extension of z combines with low noise to imply S is an admissible recurrence. Corollary 20.12 implies S has a unique solution, U, which satisfies the strong Akra-Bazzi condition relative to S and z^*. Since the recursion set of S is unbounded, the weak Akra-Bazzi condition is also satisfied, i.e.,

$$U(x) = \Theta\left(x^p \left(1 + \int_{x_0}^{x} \frac{z^*(u)}{u^{p+1}} \, du \right) \right)$$

where $x_0 = \inf I$. Since z^* is identically zero,

$$U(x) = \Theta(x^p).$$

The incremental cost, g, of R is a non-negative real-valued function on the (unbounded) recursion set I by definition of a divide-and-conquer recurrence, so $g = \Omega(z)$. Since z is bounded on bounded sets, Lemma 29.1 implies $T = \Omega(U)$. Therefore,

$$T(x) = \Omega(x^p)$$

as claimed. □

Corollary 29.4. Let R be a divide-and-conquer recurrence that has low noise and an unbounded recursion set that contains only integers. If T is the solution of R, then

$$T(n) = \Omega(n^p)$$

where p is the Akra-Bazzi exponent of R.

Proof. Lemma 21.1 implies R satisfies the bounded depth condition and has a unique solution T as implicitly claimed. The proposition follows from Corollary 29.3. □

Definition. A real-valued function f on a set S of real numbers is *asymptotically locally* $\Theta(1)$ if $\sup S = \infty$ and the restriction of f to some non-empty upper subset of S is locally $\Theta(1)$.

Of course, every locally $\Theta(1)$ function on a set S of real numbers with $\sup S = \infty$ is asymptotically locally $\Theta(1)$ because S is an upper subset of itself and is non-empty ($\sup \emptyset = -\infty$).

We now consider the effect of relaxing the requirements that the base case is $\Theta(1)$ and the incremental cost is non-negative.

Lemma 29.5. Let

$$R = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k)$$

be a divide-and-conquer recurrence that satisfies the bounded depth condition and has low noise and an unbounded recursion set I. Assume the incremental cost, g, is bounded on bounded sets.

Let $r_1, \dots, r_k: I \to D$ be the dependencies of R, i.e., $r_i(z) = b_i z + h_i(z)$ for all $z \in I$ and all $i \in \{1, \dots, k\}$. Let T be the solution of R.

Let f^* and g^* be real-valued functions on $D\setminus I$ and I respectively. Define a real-valued function T^* on D by the recurrence

$$T^*(x) = \begin{cases} f^*(x), & \text{for } x \in D\backslash I \\ \sum_{i=1}^{k} a_i T^*\big(r_i(x)\big) + g^*(x), & \text{for } x \in I. \end{cases}$$

If g^* is asymptotically non-negative with $g^* = \Theta(g)$, and T^* is asymptotically locally $\Theta(1)$, then $T^* = \Theta(T)$.

Proof. Lemma 9.4 implies R has a unique solution, T, as implicitly claimed. Furthermore, T is locally $\Theta(1)$. Finite recursion of R implies finite recursion of the second recurrence, which has a unique solution T^* by Lemma 8.2.

Corollary 9.9 implies there exists a non-empty upper subset Y of I and real numbers $0 < \alpha \leq \beta < 1$ such that

$$\alpha y \leq r_i(y) \leq \beta y$$

for all $y \in Y$ and all $i \in \{1, \dots, k\}$. The set Y is an upper subset of D because I is an upper subset of D.

By hypothesis, there exists a non-empty upper subset U of D such that the restriction of T^* to U is locally $\Theta(1)$. There also exists a non-empty upper subset W of I and positive real numbers λ_1 and λ_2 such that

$$\lambda_1 g(z) \leq g^*(z) \leq \lambda_2 g(z)$$

for all $z \in W$, so g^* is bounded on bounded subsets of W. Observe that W is also an upper subset of D. Define

$$E = Y \cap U \cap W,$$

so E is a non-empty upper subset of D. Furthermore, E is contained in the subset I of D, so E is an upper subset of I. Let

$$J = E \cap \left(\frac{\inf E}{\alpha}, \infty\right),$$

so J is a non-empty upper subset of E and I. Furthermore,

$$\inf J > \inf E \geq \inf I > 0$$

and $\sup J = \sup I = \infty$. Observe that

$$\inf(E\backslash J) \geq \inf E > 0.$$

Furthermore,

$$\sup(E\backslash J) \leq \inf J < \infty$$

because J is a non-empty upper subset of E. Therefore, T and T^* are $\Theta(1)$ on $E\backslash J$. Also observe that $r_i(E) \subseteq r_i(I) \subseteq D$ and

$$r_i(J) \subseteq r_i(E) \cap (\inf E, \infty) \subseteq D \cap (\inf E, \infty) = E \cap (\inf E, \infty) \subseteq E$$

for each index i, so E contains $r_i(J)$ for each such i. Therefore,

$$Q = \left(E, J, a_1, \dots, a_k, b_1, \dots, b_k, T|_{E\setminus J}, g|_J, h_1|_J, \dots, h_k|_J\right)$$

and

$$S = \left(E, J, a_1, \dots, a_k, b_1, \dots, b_k, T^*|_{E\setminus J}, g^*|_J, h_1|_J, \dots, h_k|_J\right)$$

are divide-and-conquer recurrences that satisfy the ratio condition. Lemma 9.6 implies Q and S satisfies the bounded depth condition and have unique solutions, which are $T|_E$ and $T^*|_E$ by inspection. Lemma 29.1 implies $T^*|_E = \Theta(T|_E)$, i.e., $T^* = \Theta(T)$. □

We now give an interpretation of Lemma 29.5 for integer recurrences:

Corollary 29.6. Let

$$R = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k)$$

be a divide-and-conquer recurrence that has low noise and an unbounded recursion set I that contains only integers.

Let $r_1, \dots, r_k: I \to D$ be the dependencies of R, i.e., $r_i(z) = b_i z + h_i(z)$ for all $z \in I$ and all $i \in \{1, \dots, k\}$. Let T be the solution of R.

Let f^* and g^* be real-valued functions on $D\setminus I$ and I respectively. Define a real-valued function T^* on D by the recurrence

$$T^*(n) = \begin{cases} f^*(n), & \text{for } n \in D\setminus I \\ \displaystyle\sum_{i=1}^{k} a_i T^*\big(r_i(n)\big) + g^*(n), & \text{for } n \in I. \end{cases}$$

If g^* is asymptotically non-negative with $g^* = \Theta(g)$, and T^* is asymptotically positive, then $T^* = \Theta(T)$.

Proof. Lemma 21.1 implies R satisfies the bounded depth condition and has a unique solution T as implicitly claimed. Finite recursion of R implies finite recursion of the second recurrence, which has a unique solution T^* by Lemma 8.2.

Each bounded subset S of the domain, I, of g is a finite set of integers, so $g(S)$ is a finite set of real numbers and is therefore bounded. Since I is a non-empty upper subset of D, asymptotic positivity of T^* implies there exists a non-empty upper subset J of I such that the restriction of T^* to J is positive. Each bounded subset W of J is finite, so $T^*(W)$ is a finite set of positive real numbers, which implies T^* is $\Theta(1)$ on each such W, i.e., the restriction of T^* to J is locally $\Theta(1)$. Therefore, T^* is asymptotically locally $\Theta(1)$. The proposition follows from Lemma 29.5. □

Bounded recursion sets. For sake of completeness, we now provide an adaptation of Lemma 29.1 for recurrences with bounded recursion sets. Of course, asymptotic relationships (at $+\infty$) between solutions and between incremental costs are meaningless when the recursion set is bounded. Lemma 29.1 refers to boundedness of incremental costs on bounded sets. With a bounded recursion set, that property is equivalent to global boundedness of incremental costs.

Lemma 29.7. Let Q and R be divide-and-conquer recurrences that satisfy the bounded depth condition and have bounded recursion sets. Let D_Q, g_Q, S and D_R, g_R, T be the domains, incremental costs, and solutions of Q and R, respectively. Then:

(1) If g_Q is bounded, then there exists a positive real number μ such that $S(x) \leq \mu T(x)$ for all $x \in D_Q \cap D_R$.

(2) If g_R is bounded, then there exists a positive real number λ such that $\lambda T(x) \leq S(x)$ for all $x \in D_Q \cap D_R$.

(3) If g_Q and g_R are bounded, then there exist positive real numbers λ and μ such that $\lambda T(x) \leq S(x) \leq \mu T(x)$ for all $x \in D_Q \cap D_R$.

Proof. Corollary 8.5 implies Q and R have unique solutions S and T, respectively, as implicitly claimed. Let I_Q and I_R be the recursion sets of Q and R respectively.

We now prove (1). Suppose g_Q is bounded. Corollary 9.4 implies S is locally $\Theta(1)$ and $T(I_R)$ has a positive lower bound y.

By definition, I_R is non-empty, so $T(I_R)$ is a non-empty set of real numbers, which implies y is finite, i.e., real; furthermore, $T(D_R \backslash I_R)$ has a positive lower bound z. Of course, finite recursion of R implies $D_R \backslash I_R$ is non-empty, so $T(D_R \backslash I_R)$ is non-empty and z is finite, i.e., real. Let α be the minimum of y and z, so α is a positive real lower bound for T.

By definition, I_Q is a non-empty upper subset of D_Q, so

$$\sup D_Q = \sup I_Q < \infty.$$

Then Lemma 9.1 implies $S = \Theta(1)$, so S has a finite, i.e., real upper bound β. (Recall our definition in Section 1 of $\Theta(1)$ on a set with a finite upper bound.). The domain D_Q of S contains I_Q and is therefore also non-empty, so positivity of S implies $\beta > 0$.

Define the positive real number

$$\mu = \frac{\beta}{\alpha},$$

so

$$S(x) \le \beta \le \beta \cdot \frac{T(x)}{\alpha} = \mu T(x)$$

for all $x \in D_Q \cap D_R$. Part (1) is proved.

We now prove (2). Suppose g_R is bounded. Part (1) implies the existence of a positive real number δ such that

$$T(x) \le \delta S(x)$$

for all $x \in D_Q \cap D_R$. Then

$$\lambda T(x) \le S(x)$$

for all such x where λ is the positive real number $1/\delta$. Part (2) is proved.

Finally, (3) follows from (1) and (2). □

Lemma 29.7 has a simple interpretation for integer recurrences:

Corollary 29.8. Let Q and R be divide-and-conquer recurrences with bounded recursion sets that contain only integers. Let D_Q, S and D_R, T be the domains and solutions of Q and R, respectively. Then there exist positive real numbers λ and μ such that

$$\lambda T(x) \le S(x) \le \mu T(x)$$

For all $x \in D_Q \cap D_R$.

Proof. Lemma 21.1 says Q and R satisfy the bounded depth condition; furthermore, Q and R have unique solutions S and T, respectively, as implicitly claimed. The recursion sets of Q and R are finite because they are bounded and contain only integers. Therefore, the incremental costs of Q and R have finite ranges, which implies the incremental costs are bounded. The proposition follows from Lemma 29.7. □

30. Noise Bounds

By definition, an admissible recurrence has low noise. If the recursion set is unbounded, Lemma 20.1 implies the recurrence satisfies Leighton's noise condition relative to some $\varepsilon > 0$ on some non-empty upper subset J of the recursion set, i.e.

$$|h_i(x)| \leq \frac{x}{\log^{1+\varepsilon} x}$$

for all $x \in J$ and all $i \in \{1, \dots, k\}$ where $h_1, \dots, h_k$ are the noise terms. Theorem 2 in [Le] assumes satisfaction of Leighton's noise condition on the entire recursion set.

A remark at the end of [Le] says "It is worth noting that the $x/\log^{1+\varepsilon} x$ limit on the size of $|h_i(x)|$ is nearly tight, since the solution of the recurrence

$$T(x) = \begin{cases} \Theta(1), & \text{for } 1 \leq x \leq x_0 \\ 2T\left(\dfrac{x}{2} + \dfrac{x}{\log x}\right), & \text{for } x > x_0 \end{cases}$$

is

$$T(x) = x \log^{\Theta(1)} x,$$

which is different than the solution of $\Theta(x)$ for the recurrence without the $x/\log x$ term." The condition $x_0 \geq 1$ is obviously intended.

The semi-divide-and-conquer recurrence above violates our definition of low noise and is therefore inadmissible. If $x_0 \geq 2$, the recurrence without the $x/\log x$ term is an admissible divide-and-conquer recurrence that satisfies the ratio condition; Corollary 20.13 implies existence of a unique solution T, which satisfies the strong Akra-Bazzi condition; the Akra-Bazzi exponent is 1 and the incremental cost is 0, so, $T(x) = \Theta(x)$ as claimed. If $x_0 \in [1, 2)$, the recurrence without the $x/\log x$ term is ill posed with no solution: there exists $y \in (x_0, 2)$; the recurrence says $T(y) = 2T(y/2)$, but $y/2$ is not contained in $[1, \infty)$, which is the recurrence's intended domain.

We shall consider the interpretation of Leighton's asserted solution to the original recurrence and prove its validity when $x_0 > e^2$ (i.e., the recurrence is proper). There is more to the story when $x_0 \le e^2$. For the remainder of this section, let

$$\alpha = e^{-1+\sqrt{3}} \approx 2.07934$$

and define the function

$$B: (1,\infty) \to (1,\infty)$$

by

$$B(x) = \frac{x}{2} + \frac{x}{\log x}.$$

The implicit assertion that $B(x) > 1$ for all $x > 1$ is easily justified: The quantities $x/2$ and $x/\log x$ are positive for each such x, so $B(x) > \max(x/2, x/\log x)$. If $x \ge 2$, then $x/2 \ge 1$. If $1 < x < 2$, then $x/\log x > x/\log 2 > x > 1$.

We list some simple facts about B:

Lemma 30.1.

(1) $B(e^2) = e^2$, $B(x) > x$ for all $x \in (1, e^2)$, and $B(y) < y$ for all $y \in (e^2, \infty)$.

(2) $B|_{(1,\alpha]}$ is strictly decreasing, and $B|_{[\alpha,\infty)}$ is strictly increasing. In particular, $B(x) \ge B(\alpha) > \alpha$ for all $x \in (1,\infty)$.

(3) The intervals $[t, e^2)$, (t, e^2) and (u, ∞) are B-invariant for all $t \in [\alpha, e^2)$ and all $u \in [1, e^2]$.

(4) For all $x > 1$,
$$\lim_{n\to\infty} B^n(x) = e^2.$$

Proof. (1) follows from $\log e^2 = 2$, $\log x < 2$ for all $x \in (1, e^2)$, and $\log y > 2$ for all $y > e^2$. The derivative of B is

$$\frac{\log^2 x + 2\log x - 2}{2\log^2 x},$$

which has α as its only root (in the domain of B). Since α is the unique critical point of B, and
$$\lim_{x\to 1^+} B(x) = \lim_{x\to\infty} B(x) = \infty,$$

we conclude that $B|_{(1,\alpha]}$ is strictly decreasing, and $B|_{[\alpha,\infty)}$ is strictly increasing, so $B(x) \ge B(\alpha)$ for all $x \in (1,\infty)$. (1) implies $B(\alpha) > \alpha$. Thus (2) holds.

If $\alpha \le t \le x < e^2$, then (1) and (2) imply $x < B(x) < B(e^2) = e^2$, so the intervals $[t, e^2)$ and (t, e^2) are B-invariant. Suppose $u \in [1, e^2]$ and $y > u$. If $y > e^2$, then (1) and (2) imply $B(y) > B(e^2) = e^2 \geq u$. If instead $y \leq e^2$, then (1) implies $B(y) \geq y > u$. Thus (u, ∞) is B-invariant, and (3) holds.

B-invariance of $(1, \infty)$ implies the real-valued function B^n is defined on $(1, \infty)$ for all non-negative integers n. The exponent n refers to composition of functions, not exponentiation of function values, and B^0 is the identity map on $(1, \infty)$.

If $x \geq \alpha$, then (1) and (3) imply the sequence

$$x, B(x), B^2(x), B^3(x), \ldots$$

is increasing if $x < e^2$, stationary if $x = e^2$, and decreasing if $x > e^2$. In particular, the sequence is monotonic. By (3), the sequence is contained in the closed and bounded interval $I = [\min\{x, e^2\}, \max\{x, e^2\}]$. The sequence converges to a limit contained in I, which is contained in the domain of B. Continuity of B implies

$$B\left(\lim_{n\to\infty} B^n(x)\right) = \lim_{n\to\infty} B^{n+1}(x) = \lim_{n\to\infty} B^n(x).$$

Thus (1) implies

$$\lim_{n\to\infty} B^n(x) = e^2.$$

Finally, suppose $1 < x < \alpha$, so $B(x) > \alpha$ by (2), and

$$\lim_{n\to\infty} B^n(x) = \lim_{n\to\infty} B^{n+1}(x) = \lim_{n\to\infty} B^n\big(B(x)\big) = e^2.$$

□

Infinite recursion. Suppose $1 \leq x_0 \leq e^2$. The interval (x_0, ∞) is B-invariant by Lemma 30.1(3). (In particular, the recurrence has infinite depth of recursion at each $x > x_0$.) We may define a solution of the recurrence with $T(x) = 0$ for all $x > x_0$. The restriction of T to $[1, x_0]$ can be any function that is $\Theta(1)$. Observe that

$$T(x) \neq x \log^{\Theta(1)} x.$$

The next proposition establishes the existence of other solutions when $1 \leq x_0 \leq e^2$. However, all solutions satisfy

$$T(e^2) = 2T\big(B(e^2)\big) = 2T(e^2),$$

which implies $T(e^2) = 0$.

The proof of the next proposition is similar to Section 13.

Lemma 30.2. Suppose $1 \leq x_0 \leq e^2$ and $f{:}[1, x_0] \to \boldsymbol{R}$. For each real-valued function g on (x_0, ∞), there exists a function $T{:}\,[1, \infty) \to \boldsymbol{R}$ such that $T|_{[1,x_0]} = f$,

$$T(x) = 2T\left(\frac{x}{2} + \frac{x}{\log x}\right)$$

for all $x > x_0$, and T agrees with g on some unbounded set.

Proof. Define $\gamma = \max\{\alpha, x_0\}$, so $\gamma \in [\alpha, e^2]$. Let $I = (\gamma, \infty)$ and $S = I\backslash\{e^2\}$. Observe that $S \subset domain(B)$. Define $\beta = B|_S$. If $\gamma = e^2$, then $S = (e^2, \infty)$; if $\gamma \neq e^2$, then $\gamma \in [\alpha, e^2)$ and

$$S = (\gamma, e^2) \cup (e^2, \infty).$$

Lemma 30.1(3) implies S is B-invariant and is therefore β-invariant. Thus $\beta^n: S \to S$ is defined for all non-negative integers n. The exponent n refers to composition of functions, not exponentiation of function values, and β^0 is the identity map on S.

Lemma 30.1(2) implies the restriction of B to $[\alpha, \infty)$ is increasing and is therefore injective. Then β is injective since $S \subset [\alpha, \infty)$. For each non-negative integer n, the function β^n is injective and has an inverse $(\beta^n)^{-1}: \beta^n(S) \to S$. Define functions β_n for all integers n by $\beta_n = \beta^n$ when $n \geq 0$, and $\beta_n = \left(\beta^{|n|}\right)^{-1}$ when $n < 0$. The function β_n is injective for every integer n.

(If $x_0 < e^2$, i.e., $\gamma < e^2$, the function β is not surjective, so the domain of β^{-1} is properly contained in S, which is the range of β^{-1}. Thus $\beta^{-1} \circ \beta^{-1}$ is undefined, i.e., $(\beta^{-1})^2$ is undefined. Indeed, $(B^{-1})^n$ is undefined for all $n > 1$. Thus we do not use the notation β^{-n} for $(\beta^n)^{-1}$.)

Define a binary relation $\sim$ on S by $y \sim z$ if there exists an integer k such that y in the domain of β_k, and $\beta_k(y) = z$, which implies z is in the domain of β_{-k}, and $\beta_{-k}(z) = y$. In other words, $\sim$ is symmetric. The relation $\sim$ is also reflexive because β_0 is the identity map on S. Now suppose $s_1, s_2, s_3 \in S$ such that $s_1 \sim s_2$ and $s_2 \sim s_3$. There exist integers m and n such that $\beta_m(s_1) = s_2$ and $\beta_n(s_2) = s_3$. (In particular, s_1 and s_2 are in the domains of β_m and β_n, respectively.) Then s_1 is in the domain of β_{m+n}, and $\beta_{m+n}(s_1) = s_3$, so $\sim$ is transitive. Therefore, $\sim$ is an equivalence relation.

Given $t \in S$, Lemma 30.1(3) implies the equivalence class of t is contained in either (γ, e^2) or (e^2, ∞). (Of course, $(\gamma, e^2) = \emptyset$ if $\gamma = e^2$). If $i, j \in \boldsymbol{Z}$ such that $\beta_i(t)$ and $\beta_j(t)$ are defined, Lemma 30.1(1) implies $\beta_i(t) = \beta_j(t)$ if and only if $i = j$.

We claim that for each transversal L of $\sim$ (i.e., $L \subseteq S$ and L contains exactly one element of each equivalence class), and each real-valued function $\lambda: L \to \boldsymbol{R}$, there exists a function $T_\lambda: [1, \infty) \to \boldsymbol{R}$ that satisfies $T_\lambda|_{[1, x_0]} = f$, $T_\lambda|_L = \lambda$, and

$$T_\lambda(x) = 2T_\lambda\left(\frac{x}{2} + \frac{x}{\log x}\right)$$

for all $x > x_0$, i.e. λ has an extension to a solution of the recurrence: Each element of S has a unique representation of the form $\beta_n(u)$ with $u \in L$ and $n \in \boldsymbol{Z}$. Define

$$T\big(\beta_n(u)\big) = \frac{\lambda(u)}{2^n}$$

for each such u and n. Observe that $\beta_n(u) \in S = domain(\beta)$, so $u \in domain(\beta_{n+1})$ and

$$T_\lambda\big(\beta_n(u)\big) = 2 \cdot \frac{\lambda(u)}{2^{n+1}} = 2T_\lambda\big(\beta_{n+1}(u)\big) = 2T_\lambda\left(B\big(\beta_n(u)\big)\right)$$

as required. Define $T_\lambda(e^2) = 0$. Lemma 30.1(1) says $B(e^2) = e^2$, so

$$T(e^2) = 2T_\lambda\big(B(e^2)\big).$$

We have defined the restriction of T_λ to $I = S \cup \{e^2\}$. If there exists $w > x_0$ such that $w \notin I$, then $\gamma = \alpha$, i.e, $I = (\alpha, \infty)$, so $B(w) \in I$ by Lemma 30.1(2). Define $T_\lambda(w) = 2T_\lambda\big(B(w)\big)$ for each such w. We have constructed a function T_λ with the required properties. (Furthermore, T_λ is uniquely determined.)

In particular, for each transversal L of $\sim$ there exists a solution T of the recurrence with $T|_{[1,x_0]} = f$ and $T|_L = g|_L$. We shall prove the lemma by showing the existence of an unbounded transversal of $\sim$.

Define an *independent set* to be any subset of S that does not contain more than one element of any equivalence class. The collection of independent sets is partially ordered by inclusion. A maximal independent set q must be a transversal. Otherwise, q contains no elements of the equivalence class of some $s \in S$. Then $q \cup \{s\}$ is an independent set that properly contains q, in contradiction of q's maximality. (Furthermore, all transversals are maximal independent sets.)

We claim that the union of a chain (a set totally ordered by inclusion) of independent sets is also an independent set: Define

$$U(C) = \bigcup_{Y \in C} Y$$

for each chain C of independent sets. Suppose x_1 and x_2 are distinct elements of $U(C)$, so there exist $c_1, c_2 \in C$ such that $x_1 \in c_1$ and $x_2 \in c_2$. Since C is a chain, either $c_1 \subseteq c_2$ or $c_2 \subseteq c_1$. Thus either c_1 or c_2 contains both x_1 and x_2, which implies x_1 and x_2 are in different equivalence classes. Therefore, $U(C)$ is an independent set containing all elements of C.

Suppose r is an independent set, and let X be the collection of independent sets containing r. Since every chain $D \subseteq X$ has an upper bound $U(D) \in X$, Zorn's lemma implies X contains a maximal element X^*. Since $r \subseteq X^*$, any independent set containing

X^* must also be an element of X. Therefore X^* is a maximal independent set, i.e., X^* is a transversal. In other words, every independent set is contained in a transversal of $\sim$.

Let A be the set of non-empty, finite, independent subsets p of S with the property $\max p > |p|$. The subset $\{s\}$ of S is non-empty, finite, and independent for each $s \in S$; furthermore,

$$\max\{s\} = s > 1 = |\{s\}|,$$

so $\{s\} \in A$. Therefore, A is non-empty. Since each $p \in A$ is finite and each equivalence class is countable, the set

$$\overline{p} = \bigcup_{x \in p} x^{\sim}$$

is countable where $x^{\sim}$ denotes the equivalence class of x. For each $p \in A$ there exists v in the uncountable subset

$$(\max\{e^2, 1 + \max p\}, \infty)$$

of S such that $v \notin \overline{p}$. The non-empty, finite, independent set $p^* = p \cup \{v\}$ properly contains p. Furthermore,

$$\max p^* = v > 1 + \max p > 1 + |p| = |p^*|.$$

We conclude that $p^* \in A$. In particular, A has no maximal elements.

If E is a non-empty, finite chain of elements of A, then E has a maximum element. Since A has no maximal elements, there exists $b \in A$ that properly contains $\max E$. The set $E \cup \{b\}$ is a chain of elements of A that properly contains E. Since A is non-empty and singleton subsets of A are chains, the empty set is not a maximal chain in A either. Therefore, all maximal chains of elements of A are infinite.

The Hausdorff maximal principle implies the existence of a maximal chain A^* of elements of A. Since A^* is a chain of finite sets, no two distinct elements of A^* have the same cardinality. Because A^* is infinite,

$$\sup_{a \in A^*} |a| = \infty.$$

We conclude from $\max a > |a|$ for all $a \in A$ that

$$\sup_{a \in A^*} (\max a) = \infty,$$

i.e., $\sup U(A^*) = \infty$. The unbounded independent set $U(A^*)$ can be extended to a transversal of $\sim$, which is also unbounded. □

Exponential example. Let $x_0 \in [1, e^2]$ and let $f: [1, x_0] \to \boldsymbol{R}$ be $\Theta(1)$ for conformity with Leighton's example and our definition of a divide-and-conquer recurrence. Lemma 30.2 implies there exists a solution T of the recurrence

$$T(x) = \begin{cases} f(x), & \text{for } 1 \le x \le x_0 \\ 2T\left(\dfrac{x}{2} + \dfrac{x}{\log x}\right), & \text{for } x > x_0 \end{cases}$$

that agrees with e^x on some unbounded set. In particular,

$$T(x) \ne x \log^{\Theta(1)} x.$$

Lemma 30.3. Suppose $x_0 > e^2$ and $f:[1, x_0] \to \boldsymbol{R}^+$ is $\Theta(1)$. There exists exactly one function $T: [1, \infty) \to \boldsymbol{R}$ such that $T|_{[1,x_0]} = f$ and

$$T(x) = 2T\left(\frac{x}{2} + \frac{x}{\log x}\right)$$

for all $x > x_0$. Furthermore, there exists an asymptotically positive, real-valued function $\lambda: (1, \infty)\backslash\{e\} \to \boldsymbol{R}$ such that λ is $\Theta(1)$ and

$$T(x) = x \log^{\lambda(x)} x$$

for all $x \in (1, \infty)\backslash\{e\}$.

Proof. Lemma 30.1(4) implies the divide-and-conquer recurrence (proper by Lemma 30.1(1)) is finitely recursive. Corollary 8.5 implies the recurrences has a unique solution, T, which is positive.

Since T is positive, we may define $\lambda: (1, \infty)\backslash\{e\} \to \boldsymbol{R}$ by

$$\lambda(x) = \frac{\log(T(x)/x)}{\log \log x},$$

so that

$$T(x) = x \log^{\lambda(x)} x$$

for all $x \in (1, \infty)\backslash\{e\}$. We will show that λ is asymptotically positive and $\lambda(x) = \Theta(1)$. Let $x > x_0$, so $x > e^2$. Lemma 30.1(3) implies $B^k(x) > e^2$ for each non-negative integer k. (As before, the exponent k refers to composition of functions, not exponentiation of function values, and B^0 is the identity map on $(1, \infty)$.) In particular, $B^k(x)$ is in the domain of λ for all such k. Furthermore,

$$\lambda(x) \log \log x = \log\left(\frac{2T\big(B(x)\big)}{x}\right) = \log\left(\frac{2B(x)}{x} \log^{\lambda(B(x))} B(x)\right)$$

$$= \log\left(1 + \frac{2}{\log x}\right) + \lambda\big(B(x)\big) \log\log B(x)$$

$$= \lambda\big(B^n(x)\big) \log\log B^n(x) + \sum_{k=0}^{n-1} \log\left(1 + \frac{2}{\log B^k(x)}\right),$$

where $n > 0$ is the depth of recursion at x, i.e., $B^n(x) \leq x_0$ and $B^k(x) > x_0$ for all $k \in \{0, \dots, n-1\}$. Observe that $B^n(x) \in (e^2, x_0]$.

Since f is $\Theta(1)$ and T agrees with f on $(e^2, x_0]$, there exist $z_1, z_2 \in \boldsymbol{R}^+$ such that $z_1 < T(v) < z_2$ for all $v \in (e^2, x_0]$, so $z_1/x_0 < T(v)/v < z_2/e^2$ and

$$a < \log\left(\frac{T(v)}{v}\right) < b$$

for each such v where $a = \log(z_1/x_0)$ and $b = \log(z_2/e^2)$. All $v \in (e^2, x_0]$ are in the domain of λ with $a < \lambda(v) \log\log v < b$. In particular,

$$a < \lambda\big(B^n(x)\big) \log\log B^n(x) < b.$$

For each integer $k \geq 0$, the quantity $\log B^k(x)$ is positive (indeed, $\log B^k(x) > 2$), so

$$\log\left(1 + \frac{2}{\log B^k(x)}\right) < \frac{2}{\log B^k(x)}.$$

Thus

$$a + \sum_{k=0}^{n-1} \log\left(1 + \frac{2}{\log B^k(x)}\right) \; < \; \lambda(x) \log\log x \; < \; b + 2\sum_{k=0}^{n-1} \frac{1}{\log B^k(x)}.$$

Let

$$c = \left(\frac{1}{2} + \frac{1}{\log x_0}\right)^{-1},$$

so $c > 1$. Then

$$\frac{B(w)}{w} = \frac{1}{2} + \frac{1}{\log w} \in \left(\frac{1}{2}, \frac{1}{c}\right)$$

for all $w > x_0$, i.e., $cB(w) < w < 2B(w)$. We conclude from $B^j(x) > x_0$ for all $j \in \{0, .., n-1\}$ that

$$c^{n-k} < c^{n-k} e^2 < c^{n-k} B^n(x) < B^k(x) < 2^{n-k} B^n(x) \leq 2^{n-k} x_0$$

for $0 \leq k < n$. Therefore,

$$\sum_{k=0}^{n-1} \frac{1}{\log B^k(x)} < \sum_{k=0}^{n-1} \frac{1}{(n-k)\log c} = \sum_{k=1}^{n} \frac{1}{k \log c} \le \frac{1}{\log c} \cdot \left(1 + \int_1^n \frac{dt}{t}\right) = \frac{1+\log n}{\log c}$$

and

$$\sum_{k=0}^{n-1} \log\left(1 + \frac{2}{\log B^k(x)}\right) > \sum_{k=0}^{n-1} \log\left(1 + \frac{2}{\log(2^{n-k} x_0)}\right) = \sum_{k=1}^{n} \log\left(1 + \frac{2}{\log(2^k x_0)}\right)$$

$$= \sum_{k=1}^{n} \log\left(1 + \frac{2}{k \log 2 + \log x_0}\right) > \sum_{k=1}^{n} \log\left(1 + \frac{2}{(k+1)\log x_0}\right)$$

$$= \sum_{k=2}^{n+1} \log\left(1 + \frac{2}{k \log x_0}\right) > \sum_{k=2}^{n+1} \left(\frac{2}{k \log x_0} - \frac{2}{k^2 \log^2 x_0}\right)$$

$$> \frac{2}{\log x_0} \int_2^{n+2} \frac{dt}{t} - \frac{2}{\log^2 x_0} \int_1^{n+1} \frac{dt}{t^2}$$

$$= \frac{2\log(n+2) - 2\log 2}{\log x_0} - \frac{2}{\log^2 x_0} \cdot \left(1 - \frac{1}{n+1}\right)$$

$$> \frac{2\log n - 2\log 2}{\log x_0} - \frac{2}{\log^2 x_0}.$$

Recall that $c^n < x < 2^n x_0$, which implies

$$\frac{\log x - \log x_0}{\log 2} < n < \frac{\log x}{\log c}.$$

Therefore,

$$\lambda(x) \log\log x < b + 2\left(\frac{1 + \log\left(\frac{\log x}{\log c}\right)}{\log c}\right) = b + 2\left(\frac{1 + \log\log x - \log\log c}{\log c}\right).$$

The assumption $x > x_0 > e^2$ implies $\log\log x > 0$, so

$$\lambda(x) < \frac{2}{\log c} + \frac{b \log c + 2 - 2\log\log c}{(\log c)(\log\log x)} = O(1).$$

Furthermore,

$$\lambda(x) \log\log x \; > \; a \; + \; \frac{2\log\left(\frac{\log x - \log x_0}{\log 2}\right) - 2\log 2}{\log x_0} - \frac{2}{\log^2 x_0} \; = \; \Theta(\log\log x),$$

which implies λ is asymptotically positive, and $\lambda(x) = \Omega(1)$. Therefore, $\lambda(x) = \Theta(1)$. □

31. Bounded Gap Ratios

Bounded gap ratios will play a role in our treatment of almost increasing functions.

Definition. A set S of positive real numbers has *bounded gap ratios* if there exists a real number $\alpha > 1$ such that $S \cap [x, \alpha x]$ is non-empty for all $x \in (\inf S, \sup S)$.

The definition above is satisfied for all $\alpha > 1$ when $(\inf S, \sup S)$ is contained in S. In particular, the definition is vacuously satisfied when $(\inf S, \sup S)$ is empty, i.e., S is either empty or a singleton:

$$(\inf \emptyset, \sup \emptyset) = (+\infty, -\infty) = \emptyset$$

and

$$(\inf S, \sup S) = (y, y) = \emptyset$$

when S has a single element y.

We require $\alpha \neq 1$ for convenience in the next section. In particular, the dynamic range notation Ψ_α is defined when $\alpha > 1$.

Lemma 31.1. If S is a set of real numbers with a positive lower bound and finite upper bound, then S has bounded gap ratios.

Proof. Since the empty set and positive singletons have bounded gap ratios, we may assume S is neither empty nor a singleton. Define the real number

$$\alpha = \frac{\sup S}{\inf S},$$

so $\alpha > 1$. If $x \in (\inf S, \sup S)$, then

$$\inf S < x < \sup S < \alpha x,$$

so $S \cap [x, \alpha x]$ contains $S \cap (x, \sup S)$, which is non-empty. Therefore, S has bounded gap ratios. □

Positive bounded set that does not have bounded gap ratios. Define a positive, decreasing sequence $t_0, t_1, t_2, \ldots$ by

$$t_n = e^{-e^n},$$

so

$$\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = \lim_{n\to\infty} e^{(e^{n+1}-e^n)} = \infty.$$

The positive set

$$S = \{t_n : n \in \boldsymbol{N}\}$$

is bounded because

$$\max S = t_0 = 1/e < \infty.$$

Let $\alpha > 1$. There exists $m \in \boldsymbol{N}$ such that

$$\frac{t_m}{t_{m+1}} > \alpha,$$

so

$$t_{m+1} < \frac{t_m}{\alpha}.$$

Let $x \in (t_{m+1}, t_m/\alpha)$. Then

$$t_{m+1} < x < \alpha x < t_m,$$

which implies $x \in (\inf S, \sup S)$ and $S \cap [x, \alpha x] = \emptyset$. Therefore, S does not have bounded gap ratios. Lemma 31.1 is inapplicable to S because $\inf S = 0$.

Lemma 31.2. If A and B are sets of positive real numbers with bounded gap ratios, then $A \cup B$ has bounded gap ratios.

Proof. Let $C = A \cup B$. We may assume $C \notin \{A, B\}$, so A and B are non-empty. Positivity and non-emptiness of A and B imply $\inf A$ and $\inf B$ are non-negative real numbers and $\sup A, \sup B \in (0, \infty]$. Without loss of generality, we may choose notation so that $\inf A \leq \inf B$.

For each set S of real numbers, define the open interval $S^* = (\inf S, \sup S)$. Define

$$M = C^* \cap [\sup A, \inf B].$$

Let $w \in C^*$, so

$$\inf A = \min\{\inf A, \inf B\} = \inf C < w < \sup C = \max\{\sup A, \sup B\}.$$

If $w < \sup A$, then $w \in A^*$. Suppose $w \geq \sup A$, so $\sup C = \sup B$ and $w < \sup B$; if $w > \inf B$, then $w \in B^*$; if $w \leq \inf B$, then $w \in M$. We conclude that $A^* \cup B^* \cup M$ contains C^*. (The reverse containment is also true, so $C^* = A^* \cup B^* \cup M$.)

If M is non-empty, then either $\sup A < \inf B$ or

$$\sup A = \inf B < \sup C.$$

Therefore, $\sup A$ and $\inf B$ are positive real numbers when M is non-empty. Define a real number $\gamma > 1$ by

$$\gamma = 2 \cdot \frac{\inf B}{\sup A}$$

if M is non-empty and $\gamma = 2$ if M is empty. Observe that $z \leq \inf B < \gamma z$ for all $z \in M$, so

$$B \cap [z, \gamma z] \neq \emptyset$$

for all such z.

By hypothesis, there exist real numbers $\alpha > 1$ and $\beta > 1$ such that $A \cap [x, \alpha x]$ is non-empty for all $x \in A^*$ and $B \cap [y, \beta y]$ is non-empty for all $y \in B^*$. Let $\lambda = \max\{\alpha, \beta, \gamma\}$, so $\lambda > 1$ is a real number. Suppose $t \in C^*$. If $t \in A^*$, then

$$\emptyset \neq A \cap [t, \alpha t] \subseteq C \cap [t, \lambda t].$$

If instead $t \in B^*$, then

$$\emptyset \neq B \cap [t, \beta t] \subseteq C \cap [t, \lambda t].$$

If $t \notin A^*$ and, $t \notin B^*$ then $t \in M$, so

$$\emptyset \neq B \cap [t, \gamma t] \subseteq C \cap [t, \lambda t].$$

Therefore, C has bounded gap ratios. □

Of course, Lemma 31.2 can be easily extended by induction to finite unions of sets with bounded gap ratios. However, we have no need for such a result.

Lemma 31.3. If R is a semi-divide-and-conquer recurrence with low noise, then the recursion set of R has bounded gap ratios.

Proof. Let D be the domain of R, so D is a set of real numbers. Let I be the recursion set of R, so I is a non-empty upper subset of D with a positive lower bound. By Lemma 31.1, we may assume $\sup I = \infty$.

Lemma 9.8 implies there exists a non-empty upper subset J of I and real numbers λ and μ with $\lambda > \mu > 1$ such that $x/\lambda < r(x) < x/\mu$ for all $x \in J$ and each dependency r of R. (In the notation of Lemma 9.8, $\lambda = 1/\alpha$ and $\mu = 1/\beta$.) Non-emptiness of J implies $\inf J < \infty$. Furthermore, $\inf J \geq \inf I > 0$ and $\sup J = \sup I$, i.e., $\sup J = \infty$. Observe that J is an upper subset of D because J is an upper subset of the upper subset I of D.

Let $t \in (\inf J, \infty)$, so

$$D \cap [t, \infty) = J \cap [t, \infty) \neq \emptyset.$$

We conclude from

$$[t,\infty)=\bigcup_{j=0}^{\infty}\left[\lambda^j t,\lambda^{j+1}t\right)$$

that

$$\left\{j\in \boldsymbol{N} : D\cap\left[\lambda^j t,\lambda^{j+1}t\right)\neq\emptyset\right\}$$

is a non-empty set of non-negative integers and therefore has a least element m. Observe that

$$[\lambda^m t,\lambda^{m+1}t)\subset[\lambda^m t,\infty)=\bigcup_{j=0}^{\infty}\left[\lambda^m\mu^j t,\lambda^m\mu^{j+1}t\right),$$

so

$$\left\{j\in \boldsymbol{N} : D\cap\left[\lambda^m\mu^j t,\lambda^m\mu^{j+1}t\right)\neq\emptyset\right\}$$

is a non-empty set of non-negative integers and therefore has a least element n. Then

$$D\cap[\lambda^m t,\lambda^m\mu^n t)=\emptyset$$

and there exists

$$u\in D\cap[\lambda^m\mu^n t,\lambda^m\mu^{n+1}t).$$

Observe that

$$u\in D\cap[t,\infty)\subseteq J\subseteq I.$$

Let s be a dependency of R, so $domain(s)=I$. In particular, $u\in domain(s)$. Furthermore, $s(u)\in D$ and

$$\lambda^{m-1}t\leq\lambda^{m-1}\mu^n t\leq\frac{u}{\lambda}<s(u)<\frac{u}{\mu}<\lambda^m\mu^n t.$$

We conclude from $s(u)<\lambda^m\mu^n t$ and

$$D\cap[\lambda^m t,\lambda^m\mu^n t)=\emptyset$$

that $s(u)<\lambda^m t$, so

$$s(u)\in D\cap[\lambda^{m-1}t,\lambda^m t),$$

which implies $m-1\notin\boldsymbol{N}$. We conclude from $m\in\boldsymbol{N}$ that $m=0$. The definition of m implies

$$D\cap[t,\lambda t)\neq\emptyset,$$

i.e.

$$J\cap[t,\lambda t]\neq\emptyset.$$

Therefore, J has bounded gap ratios.

The set $I\backslash J$ has $\inf I$ as a positive lower bound and is bounded above by the real number $\inf J$. Lemma 31.1 implies $I\backslash J$ has bounded gap ratios. Lemma 31.2 implies $J\cup(I\backslash J)$ has bounded gap ratios, i.e., I has bounded gap ratios as required. □

32. Almost Increasing Functions

The admittedly poor terminology defined below is nonstandard.

Definition. A real-valued function f on a set S of real numbers is *almost increasing* if there exists a positive real number c such that $f(x) \leq cf(y)$ for all $x, y \in S$ with $x < y$.

For example, a monotonically increasing real-valued function on a set S of real numbers is almost increasing with $c = 1$. The restriction of an almost increasing function to a subset of its domain is also almost increasing. The empty function vacuously satisfies the definition.

We start with a simple observation:

Lemma 32.1. If f is a positive, almost increasing, real-valued function on a set S of real numbers with $sup\ S = \infty$, then $f = \Omega(1)$.

Proof. The set S is non-empty because $sup\ S \neq -\infty$. Let $z \in S$. There exists a real number $c > 0$ such that $f(z) \leq cf(x)$ for all $x \in S \cap (z, \infty)$, so $f(x) \geq f(z)/c > 0$ for all such x. Then $f = \Omega(1)$ because $sup\ S = \infty$. □

Lemma 32.2. Let I be a positive, unbounded set (of real numbers) with bounded gap ratios. Suppose g is a real-valued function on I, and G is a polynomial-growth extension of g to $[\inf I, \infty)$. If g is almost increasing, then G is almost increasing.

Proof. The set I is non-empty because I is unbounded. Let $x_0 = \inf I$, so $x_0 < \infty$. Lemma 2.2(1) and polynomial growth of G imply $[x_0, \infty)$ is a positive set, i.e., $x_0 > 0$, so x_0 is a positive real number. Finiteness of x_0 and unboundedness of I imply $\sup I = \infty$. By definition of bounded gap ratios, there exists a real number $\alpha > 1$ such that $I \cap [t, \alpha t]$ is non-empty for all $t \in (x_0, \infty)$. The set $I \cap [x_0, \alpha x_0]$ is also non-empty since $x_0 = \inf I$ and $\alpha x_0 > x_0$.

Lemma 2.7 implies G is either positive or identically zero. If G is identically zero, then G is almost increasing, so we may assume G is positive and $\Psi_\alpha(G)$ is defined. Lemmas 2.10(2) and 2.16 imply $1 \leq \Psi_\alpha(G) < \infty$, so $\Psi_\alpha(G)$ is a positive real number.

By definition of an almost increasing function, there exists a positive real number c such that $g(u) \leq cg(v)$ for all with $u, v \in I$ with $u < v$. Define

$$k = \max\{\Psi_\alpha(G), c\Psi_\alpha^2(G)\},$$

so k is a positive real number.

Let $x \in [x_0, \infty)$ and $y \in (x, \infty)$. If $y \in [x, \alpha x]$, then Lemma 2.10(4) implies

$$G(x) \leq \Psi_\alpha(G)G(y) \leq kG(y).$$

Suppose instead that $y \notin [x, ax]$. Then $y > ax$. There exist $w \in I \cap [x, \alpha x]$ and $z \in I \cap [y, \alpha y]$. Observe that $w < z$, so $g(w) \leq cg(z)$, i.e., $G(w) \leq cG(z)$, which combines with Lemma 2.10(4) to imply

$$G(x) \leq \Psi_\alpha(G)G(w) \leq c\Psi_\alpha(G)G(z) \leq c\Psi_\alpha^2(G)G(y) \leq kG(y).$$

Therefore, G is almost increasing. □

Of course, Lemma 2.2(2) implies the function g of Lemma 32.2 also has polynomial growth.

33. Generalizations of the Master Theorem

In this section, we identify some circumstances under which the following assertions are satisfied by a divide-and-conquer recurrence with incremental cost g, Akra-Bazzi exponent p, and solution T:

(1) If $g(x) = O(x^{p-\varepsilon})$ for some $\varepsilon > 0$, then $T(x) = \Theta(x^p)$.

(2) If $g(x) = \Theta(x^p)$, then $T(x) = \Theta(x^p \log x)$.

(3) $g(x) = \Omega(x^{p+\varepsilon})$ for some $\varepsilon > 0$, then $T(x) = \Theta\big(g(x)\big)$.

The Master Theorem contains a similar list of three assertions and is applicable to a very narrow class of recurrences. Our Theorem 33.1 establishes validity of (1) and (2) under much more general conditions. Theorem 33.5 does the same for (3). The combination of those two theorems is a generalization of the Master Theorem. Corollaries 33.2 and 33.6 are interpretations for recurrences with recursion sets that contain only integers. They form a simpler generalization of the Master Theorem.

Theorem 33.7 is a convenient variation on (3) for admissible recurrences (regardless of whether they are proper). Corollary 33.8 is a simple interpretation for recurrences with recursion sets that contain only integers. These two propositions are not direct generalizations of the corresponding assertion of the Master Theorem.

Unbounded recursion sets. Suppose T is a solution of a semi-divide-and-conquer recurrence R with domain D, an unbounded recursion set I, and incremental cost g. The sets D and I are the domains of T and g respectively. By definition, I is a non-empty upper subset of D with a positive lower bound, so unboundedness of I implies

$$\sup D = \sup I = \infty.$$

Then asymptotic relationships are definable for the non-negative function g. If T is asymptotically non-negative, then asymptotic relationships are also definable for T according to our convention.

Theorem 33.1. Let R be a divide-and-conquer recurrence with incremental cost g and an unbounded recursion set. Suppose R has low noise and satisfies the bounded depth condition. Assume g is bounded on bounded sets. Let T be the solution of R and let p be the Akra-Bazzi exponent.

(1) If $g(x) = O(x^{p-\varepsilon})$ for some $\varepsilon > 0$, then $T(x) = \Theta(x^p)$.

(2) If $g(x) = \Theta(x^p)$, then $T(x) = \Theta(x^p \log x)$.

Proof. Corollary 8.5 implies R has a unique solution T as implicitly claimed. Furthermore, T is positive. Let I be the recursion set of R and let $x_0 = \inf I$, so $x_0 > 0$. For each non-negative real-valued function γ on I, let

$$S_\gamma = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, \gamma, h_1, \dots, h_k)$$

where

$$R = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k),$$

i.e., S_γ is the divide-and-conquer recurrence that is identical to R, except perhaps for the incremental cost, which is γ. (For example, $S_g = R$.) The recurrence S_γ also has low noise, satisfies the bounded depth condition, and has Akra-Bazzi exponent p. Corollary 8.5 implies S_γ has a unique solution U_γ, which is positive.

Suppose $g(x) = O(x^{p-\varepsilon})$ for some $\varepsilon > 0$ and define $\lambda: I \to \boldsymbol{R}$ by $\lambda(x) = x^{p-\varepsilon}$. The locally Riemann integrable function $x \mapsto x^{p-\varepsilon}$ on the positive interval $[x_0, \infty)$ has polynomial growth by Lemma 4.1(2) and is therefore a tame extension of λ. Thus S_λ is admissible. Corollary 20.12 implies

$$U_\lambda(x) = \Theta\left(x^p\left(1 + \int_{x_0}^{x} \frac{u^{p-\varepsilon}}{u^{p+1}}\,du\right)\right) = \Theta\left(x^p\left(1 + \frac{1}{\varepsilon}(x_0^{-\varepsilon} - x^{-\varepsilon})\right)\right) = \Theta(x^p).$$

Lemma 29.1(1) implies $T(x) = O\big(U_\lambda(x)\big)$, so $T(x) = O(x^p)$. Corollary 29.3 implies $T(x) = \Omega(x^p)$. Therefore, $T(x) = \Theta(x^p)$ as claimed by (1).

Now suppose instead that $g(x) = \Theta(x^p)$ and define $\mu: I \to \boldsymbol{R}$ by $\mu(x) = x^p$. The locally Riemann integrable function $x \mapsto x^p$ on the positive interval $[x_0, \infty)$ has polynomial growth by Lemma 4.1(2) and is therefore a tame extension of μ. Thus S_μ is admissible. Corollary 20.12 implies

$$U_\mu(x) = \Theta\left(x^p\left(1 + \int_{x_0}^{x} \frac{u^p}{u^{p+1}}\,du\right)\right) = \Theta\big(x^p(1 + \log x - \log x_0)\big) = \Theta(x^p \log x).$$

Lemma 29.1(3) implies $T(x) = \Theta\left(U_\mu(x)\right)$, so $T(x) = \Theta(x^p \log x)$ as claimed by (2). □

Corollary 33.2. Let R be a divide-and-conquer recurrence with low noise and an unbounded recursion set that contains only integers. Let T be the solution of R, let g be the incremental cost, and let p be the Akra-Bazzi exponent.

(1) If $g(n) = O(n^{p-\varepsilon})$ for some $\varepsilon > 0$, then $T(n) = \Theta(n^p)$.

(2) If $g(n) = \Theta(n^p)$, then $T(n) = \Theta(n^p \log n)$.

Proof. Lemma 21.1 implies R satisfies the bounded depth condition and has a unique solution T as implicitly claimed. The domain of g is the recursion set I, which contains only integers. If S is a bounded subset of I, then S is finite, so $g(S)$ is a finite set of real numbers and is therefore bounded, i.e., g is bounded on bounded sets. The proposition follows from Theorem 33.1. □

Lemma 33.3. If T is a non-negative solution of a semi-divide-and-conquer recurrence R with an unbounded recursion set I, then

$$T(x) = \Omega\big(g(x)\big)$$

where g is the incremental cost of R.

Proof. Let $r_1, \dots, r_k$ be the dependencies of R. There exist positive real numbers $a_1, \dots, a_k$ such that

$$T(x) = g(x) + \sum_{i=1}^{k} a_i T\big(r_i(x)\big)$$

for each element x of I. Non-negativity of T and $a_1, \dots, a_k$ imply $T(x) \geq g(x)$ for all such x. Then $T(x) = \Omega\big(g(x)\big)$ since g is non-negative and I is an unbounded upper subset of the domain of T. □

Corollary 33.4. If T is the solution of a finitely recursive semi-divide-and-conquer recurrence R with an unbounded recursion set, then $T(x) = \Omega\big(g(x)\big)$ where g is the incremental cost of R.

Proof. Corollary 8.5 implies R has a unique solution T as implicitly claimed. Furthermore, T is positive. The proposition follows from Lemma 33.3. □

Upper subsets. The upper-subset property is transitive: If A, B, and C are sets of real numbers such that A is an upper subset of B and B is an upper subset of C, then A is an upper subset of C.

If E is an upper subset of a set F of real numbers, and S is a set with $E \subseteq S \subseteq F$, then E is an upper subset of S.

If X and Y are upper subsets of a set W of real numbers, then either $X \subseteq Y$ or $Y \subseteq X$, so $X \cup Y$ and $X \cap Y$ are elements of $\{X, Y\}$. In particular, $X \cup Y$ and $X \cap Y$ are upper subsets of W. Furthermore, $X \cap Y$ is an upper subset of X and Y, which are upper subsets of $X \cup Y$. The set $X \cap Y$ is non-empty if X and Y are non-empty.

If I is a positive, unbounded set of real numbers, then each non-empty upper subset of I is unbounded (and positive). Each intersection of I with a positive unbounded interval is a non-empty upper subset of I. Furthermore, if H is a non-empty upper subset of I, then H is the intersection of I with the positive unbounded interval

$$(\inf H, \infty) \cup (H \cap \{\inf H\}),$$

which is either $(\inf H, \infty)$ or $[\inf H, \infty)$.

All of the aforementioned assertions about upper subsets can be easily verified by the reader.

Theorem 33.5. Let

$$R = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k)$$

be a divide-and-conquer recurrence with unbounded recursion set, I. Assume R satisfies the bounded depth condition and either

(1) The incremental cost, g, is locally $\Theta(1)$, or

(2) R has low noise, g is bounded on bounded sets, and g is asymptotically locally $\Theta(1)$.

Let T be the solution of R. Suppose

$$g(x) = \Omega(x^{p+\varepsilon})$$

for some $\varepsilon > 0$ where p is the Akra-Bazzi exponent of R. Let $r_1, \dots, r_k$ be the dependencies of R. If there exists a non-empty upper subset J of I such that $r_i(t) \in I$ for all $t \in J$ and all $i \in \{1, \dots, k\}$, and if there exists a real number $c < 1$ such that

$$\sum_{i=1}^{k} a_i g\big(r_i(t)\big) \le c g(t)$$

for all $t \in J$, then

$$T(x) = \Theta\big(g(x)\big).$$

Proof. Each of conditions (1) and (2) imply g is bounded on bounded sets. Corollary 9.4 implies R has a unique solution T (as implicitly claimed), which is locally $\Theta(1)$.

We claim there exists a non-empty upper subset I^* of I and a non-empty upper subset J^* of $I^* \cap J$ such that the restriction of g to I^* is locally $\Theta(1)$, and $r_i(v) \in I^*$ for all $v \in J^*$ and all $i \in \{1, \ldots, k\}$.

Proof of claim: If condition (1) is satisfied, then the claim is satisfied with $I^* = I$ and $J^* = J$. Now suppose instead that (2) is satisfied: There exists a non-empty upper subset E of I such that the restriction of g to E is locally $\Theta(1)$. Lemma 9.8 and low noise of R imply the existence of` a non-empty upper subset H of I and a real number $0 < \gamma < 1$ such that $r_i(w) > \gamma w$ for all $w \in H$ and each index i. Let

$$I^* = E \cap H$$

and

$$J^* = I^* \cap J \cap \left(\frac{\inf I^*}{\gamma}, \infty\right),$$

so J^* is contained in I^*, H, J, and

$$\left(\frac{\inf I^*}{\gamma}, \infty\right).$$

The set I^* is a non-empty upper subset of I because E and H are non-empty upper subsets of I. Similarly, $I^* \cap J$ is a non-empty upper subset of the positive, unbounded set I. Then $I^* \cap J$ is a positive, unbounded set, which implies J^* is a non-empty upper subset of $I^* \cap J$. The restriction of g to I^* is locally $\Theta(1)$ because I^* is contained in E. Containment of J^* in J and H implies $r_i(v) \in I$ and

$$r_i(v) > \gamma v \geq \gamma \cdot \inf J^* \geq \inf I^*$$

for all $v \in J^*$ and all $i \in \{1, \ldots, k\}$. Then $r_i(v) \in I^*$ for each such v and i since I^* is an upper subset of I. The claim is proved.

The sets I^* is an upper subset of D because I^* is an upper subset of I, which is an upper subset of D. Asymptotic behavior of T is equivalent to asymptotic behavior of the restriction of T to I^*.

The set J^* is an upper subset of I^* because J^* is an upper subset of $I^* \cap J$, which is an upper subset of I^*. The set $I^* \backslash J^*$ is bounded because

$$inf(I^* \backslash J^*) \geq inf(I^*) \geq \inf I > 0$$

and

$$sup(I^* \backslash J^*) \leq \inf J^* < \infty.$$

Then g and T are $\Theta(1)$ on $I^* \backslash J^*$. In particular, there exists positive real number α and β such that $g(y) \geq \alpha$ and $T(y) \leq \beta$ for all $y \in I^* \backslash J^*$. Then

$$T(y) \leq \frac{\beta}{\alpha} g(y)$$

for all such y. Define the positive real number

$$\lambda = \max\left\{\frac{\beta}{\alpha}, \frac{1}{1-c}\right\}.$$

Let d be the depth-of-recursion function for R relative to $D \backslash J^*$. Observe that $J^* \subseteq I^* \subseteq I$, so $J^* \subseteq I$, which implies $D \backslash I \subseteq D \backslash J^*$. Then Lemma 8.3 and finite recursion of R relative to $D \backslash I$ imply finite recursion of R relative to $D \backslash J^*$. Define

$$A = \{n \in \boldsymbol{N}: T(u) \leq \lambda g(u) \text{ for all } u \in I^* \text{ with } d(u) \leq n\}.$$

If $s \in I^*$ with $d(s) = 0$, then $s \in I^* \backslash J^*$, so

$$T(s) \leq \frac{\beta}{\alpha} g(s) \leq \lambda g(s).$$

Therefore, $0 \in A$. Let $m \in A$, and suppose $z \in I^*$ with $d(z) \leq m + 1$, so either $d(z) \leq m$ or $d(z) = m + 1$. If $d(z) \leq m$, then $T(z) \leq \lambda g(z)$. If $d(z) = m + 1$, then $d(z) > 0$, so $z \in J^*$. Then $r_i(z) \in I^*$ and $d\big(r_i(z)\big) \leq m$ for all $i \in \{1, \dots, k\}$, so

$$T\big(r_i(z)\big) \leq \lambda g\big(r_i(z)\big)$$

for each such i. Furthermore, $z \in J$ because $z \in J^* \subseteq J$. Non-negativity of $a_1, \dots, a_k$ (indeed, they are positive) implies

$$T(z) = \sum_{i=1}^{k} a_i T\big(r_i(z)\big) + g(z) \leq \lambda \sum_{i=1}^{k} a_i g\big(r_i(z)\big) + g(z) \leq (\lambda c + 1) g(z).$$

Recall that $\lambda \geq 1/(1-c)$, so $\lambda \geq \lambda c + 1$. Non-negativity of $g(z)$ implies

$$T(z) \leq \lambda g(z).$$

Therefore, $m + 1 \in A$. By induction, $A = \boldsymbol{N}$, i.e., $T(u) \leq \lambda g(u)$ for all $u \in I^*$. In particular, $T(x) = O\big(g(x)\big)$. Lemma 33.3 (or Corollary 33.4) implies $T(x) = \Omega\big(g(x)\big)$. Therefore, $T(x) = \Theta\big(g(x)\big)$. □

Corollary 33.6. Let

$$R = (D, I, a_1, \dots, a_k, b_1, \dots, b_k, f, g, h_1, \dots, h_k)$$

be a divide-and-conquer recurrence with an unbounded recursion set I that contains only integers. Assume either R has low noise or the incremental cost, g, is positive. Let T be the solution of R. Suppose

$$g(n) = \Omega(n^{p+\varepsilon})$$

for some $\varepsilon > 0$ where p is the Akra-Bazzi exponent of R. Let $r_1, \dots, r_k$ be the dependencies of R. If there exists a non-empty upper subset J of I such that $r_i(m) \in I$ for all $m \in J$ and all $i \in \{1, \dots, k\}$, and if there exists a real number $c < 1$ such that

$$\sum_{i=1}^{k} a_i g\big(r_i(m)\big) \le c g(m)$$

for all such m, then

$$T(n) = \Theta\big(g(n)\big).$$

Proof. Lemma 21.1 implies the recurrence satisfies the bounded depth condition and has a unique solution T as implicitly claimed.

The asymptotic relationship

$$g(n) = \Omega(n^{p+\varepsilon})$$

implies g is asymptotically positive.

Each bounded subset S of I is finite, so $g(S)$ is a finite set of real numbers for each such S. Therefore, g is bounded on bounded sets. If the restriction of g to a bounded subset X of I is positive, then $\inf g(X) > 0$, so g is $\Theta(1)$ on X. Asymptotic positivity of g implies g is asymptotically locally $\Theta(1)$. If g is positive, then g is locally $\Theta(1)$.

The proposition follows from Theorem 33.5. □

Theorem 33.7. Suppose T is a locally $\Theta(1)$ solution of an admissible recurrence R with an unbounded recursion set. Let g be the incremental cost of R, and let p be the Akra-Bazzi exponent. If g is positive and $g(x)/x^{p+\varepsilon}$ is almost increasing for some $\varepsilon > 0$, then $T(x) = \Theta\big(g(x)\big)$.

Proof. By definition of an admissible recurrence, R has low noise. Lemma 31.3 implies I has bounded gap ratios where I is the recursion set of R (and domain of g).

Define $x_0 = \inf I$, so $0 < x_0 < \infty$ by definition of a semi-divide-and-conquer recurrence. Unboundedness of I implies $\sup I = \infty$. Admissibility of R implies g has a tame extension H. The domain of H is an interval containing I, so $domain(H)$ contains (x_0, ∞). Let H^* be the restriction of H to

$$(x_0, \infty) \cup (I \cap \{x_0\}),$$

so the domain of H^* also contains I. The function H^* is an extension of g.

Lemma 10.1(2) implies H^* is also tame. Lemma 10.5 implies H^* can be extended to a tame function G on $[x_0, \infty)$. Observe that G is also an extension of g. (None of the aforementioned extensions are necessarily proper.) Lemma 2.7 implies G is either positive or identically zero. Positivity of g and non-emptiness of the domain, I, of g implies G is positive.

Define a positive real-valued function F on $[x_0, \infty)$ by $F(x) = G(x)/x^{p+\varepsilon}$, so F has polynomial growth by Lemma 4.1(2) and Corollary 4.4. The function F is an extension of the function $x \mapsto g(x)/x^{p+\varepsilon}$ on I, so Lemma 32.2 implies F is almost increasing. There exists a positive real number c such that $F(y) \leq cF(z)$ for all $y, z \in [x_0, \infty)$ with $y < z$.

Theorem 20.11 implies

$$T(x) = \Theta\left(x^p\left(1 + \int_{x_0}^{x} \frac{G(u)}{u^{p+1}}\,du\right)\right) = \Theta\left(x^p\left(1 + \int_{x_0}^{x} F(u)u^{\varepsilon-1}\,du\right)\right).$$

Observe that

$$\int_{x_0}^{x} F(u)u^{\varepsilon-1}\,du \leq cF(x)\int_{x_0}^{x} u^{\varepsilon-1}\,du = \frac{cF(x)}{\varepsilon}(x^\varepsilon - x_0^\varepsilon) = O(F(x)x^\varepsilon),$$

so

$$T(x) = O\big(x^p(1 + F(x)x^\varepsilon)\big).$$

Furthermore,

$$F(x)x^\varepsilon > \frac{F(x_0)x_0^\varepsilon}{c} > 0$$

for all $x > x_0$, so $F(x)x^\varepsilon = \Omega(1)$. Therefore,

$$T(x) = O(F(x)x^{p+\varepsilon}) = O\big(G(x)\big),$$

i.e., $T(x) = O\big(g(x)\big)$. Lemma 33.3 implies $T(x) = \Omega\big(g(x)\big)$, so $T(x) = \Theta\big(g(x)\big)$. □

Corollary 33.8. Suppose T is the solution of a divide-and-conquer recurrence R with low noise and an unbounded recursion set containing only integers. Let g be the incremental cost of R, and let p be the Akra-Bazzi exponent. Suppose g is positive and has polynomial growth. If $g(n)/n^{p+\varepsilon}$ is almost increasing for some $\varepsilon > 0$, then $T(n) = \Theta\big(g(n)\big)$.

Proof. Theorem 21.2 implies R is admissible and has a unique solution T as implicitly claimed. Furthermore, T satisfies the strong Akra-Bazzi condition relative to R and each tame extension of g. Theorem 20.11 implies T is locally $\Theta(1)$. The proposition follows from Theorem 33.7. □

34. Master Theorem Caveats

Section 4.5 of [CLRS] begins with:

> "The master method provides a … method for solving recurrences of the form
>
> $$T(n) = aT(n/b) + f(n)$$
>
> where … $f(n)$ is an asymptotically positive function."

On the next page, [CLRS] says "The master method depends on the following theorem" then states the following proposition:

Master Theorem. Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n),$$

where we interpret n/b to be mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

1. If $f(n) = O\left(n^{\log_b a - \varepsilon}\right)$ for some constant $\varepsilon > 0$, then $T(n) = \Theta\left(n^{\log_b a}\right)$.

2. If $f(n) = \Theta\left(n^{\log_b a}\right)$, then $T(n) = \Theta\left(n^{\log_b a} \lg n\right)$.

3. If $f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right)$ for some constant $\varepsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta\big(f(n)\big)$.

Here $\lg n$ represents the binary logarithm, $\log_2 n$, which is of course $\Theta(\log n)$.

Obvious unstated assumptions of the Master Theorem (especially in the context of other discussions of recurrences in [CLRS]). There exists a non-empty, proper, upper subset I of $\boldsymbol{N}$ such that $n/b < n$ and

$$T(n) = aT(n/b) + f(n)$$

for all $n \in I$ where n/b is an abuse of notation that represents either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$ as in the Master Theorem. For example, if the recurrence is of the form

$$T(n) = aT(\lceil n/b \rceil) + f(n),$$

then $\min I$ must be large enough that

$$(\min I)/b \le (\min I) - 1.$$

The non-empty, lower subset $\boldsymbol{N}\backslash I$ of $\boldsymbol{N}$ is the domain of the base case, which is a real valued function. The domain of f contains I. Let f^* be the restriction of f to I. (It would be convenient for the Master Theorem to specify $domain(f) = I$, i.e., $f = f^*$.) The function f^* is real-valued. With respect to part (iii) of the Master Theorem, we observe that for sufficiently large integer n, we have n and n/b (same abuse of notation) contained in I, which is contained in the domain of f.

Caveats. The Master Theorem should explicitly require that f^* is a non-negative function and the base case is a positive function. As we shall see, these assumptions are almost certainly intended by [CLRS] even though they are unstated. At the end of this section, we discuss loosening these requirements.

Our Proof of the Master Theorem. Assuming the caveats and obvious unstated assumptions listed above, recurrences described by the Master Theorem satisfy our definition of a divide-and-conquer recurrence. With the same qualification, the Master Theorem is an immediate corollary of Corollaries 33.2 and 33.6. Indeed, we recommend adoption of those propositions for general use instead of the Master Theorem, which we consider obsolete.

Non-negative $\boldsymbol{f^*}$. The statement of the Master Theorem describes f as a function but mentions no other properties of f, not even its domain or whether f is real-valued. The prelude to the Master Theorem lists only one property of f: asymptotic positivity. However, the supplied proof explicitly assumes non-negativity of f in the statements of Lemmas 4.2, 4.3, and 4.4.

Positive base case. The statements of Lemmas 4.2 and 4.4 describe the base case as $\Theta(1)$ as is common elsewhere in [CLRS]. The base case is also represented as $\Theta(1)$ in figures 4.7 and 4.8.

The subsection *Technicalities in Recurrences* [CLRS, p. 67] includes the statement

> "Boundary conditions represent another class of details that we typically ignore. … recurrences that arise from the running time of algorithms generally have $T(n) = \Theta(1)$ for sufficiently small n."

According to our definition of $\Theta(1)$ on a set of real numbers with a finite upper bound, "$T(n) = \Theta(1)$ for sufficiently small n" is equivalent to *positivity* of $T(n)$ for sufficiently small n. (We assume n is a non-negative integer as in the Master Theorem and elsewhere in [CLRS].)

[CLRS, p. 47] indicates their meaning for $\Theta(1)$ with no mention of positivity:

> "We shall often use the notation $\Theta(1)$ to indicate either a constant or a constant function…"

References to $\Theta(1)$ base cases on pages 35 and 67 of [CLRS] also refer to constants without mention of positivity. However, the supplied proof of the Master Theorem implicitly assumes the "constant" represented by $\Theta(1)$ is positive: Figures 4.7 and 4.8 assume

$$\Theta(1) * n^{\log_b a} = \Theta\left(n^{\log_b a}\right)$$

and

$$\Theta(1) * \Theta\left(n^{\log_b a}\right) = \Theta\left(n^{\log_b a}\right),$$

respectively. By definition of Θ-notation in Section 3.1 of [CLRS] the expression

$$\Theta\left(n^{\log_b a}\right)$$

represents an asymptotically positive function. Therefore, the constant value represented by the $\Theta(1)$ base case must also be positive. The aforementioned definition assumes an unbounded domain (see the first sentence of Section 3.1 of [CLRS]) and is therefore not *directly* applicable to the usage of $\Theta(1)$ in [CLRS] to describe base cases.

A recurrence that has constant, negative base case and violates conclusion of Master Theorem. Let $f: \boldsymbol{Z}^+ \to \{1\}$. Define $T: \boldsymbol{N} \to \boldsymbol{R}$ by

$$T(n) = \begin{cases} -1, & \text{if } n = 0 \\ 2T(\lfloor n/2 \rfloor) + f(n), & \text{if n} > 0, \end{cases}$$

i.e.,

$$T(n) = -1$$

for all $n \in \boldsymbol{N}$. In the notation of the Master Theorem, $a = b = 2$, so $\log_b a = 1$. Let ε be a real number satisfying $0 < \varepsilon < 1$. Observe that $1 - \varepsilon > 0$, and

$$\lim_{n \to \infty} n^{1-\varepsilon} = \infty.$$

Then

$$f(n) = O(n^{1-\varepsilon})$$

as required by part 1 of the Master Theorem, but

$$T(n) \neq \Theta(n) = \Theta\left(n^{\log_b a}\right).$$

A recurrence that has $f(1) < 0$ and violates conclusion of Master Theorem. Let $f: \boldsymbol{Z}^+ \to \boldsymbol{R}$ be the asymptotically positive function defined by

$$f(n) = \begin{cases} -3, & \text{if } n = 1 \\ 1, & \text{if } n > 1. \end{cases}$$

Define $T: \boldsymbol{N} \to \boldsymbol{R}$ by

$$T(n) = \begin{cases} 1, & \text{if } n = 0 \\ 2T(\lfloor n/2 \rfloor) + f(n), & \text{if n} > 0, \end{cases}$$

i.e.

$$T(n) = \begin{cases} 1, & \text{if } n = 0 \\ -1, & \text{if n} > 0. \end{cases}$$

Remarks from the previous example apply: In the notation of the Master Theorem, $a = b = 2$, so $\log_b a = 1$. Let ε be a real number satisfying $0 < \varepsilon < 1$. Observe that $1 - \varepsilon > 0$, and

$$\lim_{n \to \infty} n^{1-\varepsilon} = \infty.$$

Then

$$f(n) = O(n^{1-\varepsilon})$$

as required by part 1 of the Master Theorem, but

$$T(n) \neq \Theta(n) = \Theta\left(n^{\log_b a}\right).$$

Loosening the requirements for the base case and f. Corollary 29.6 allows us to replace positivity of the base case and non-negativity and asymptotic positivity of f with the requirement that T is asymptotically positive and f is asymptotically non-negative. (Caution: the notation of Corollary 29.6 conflicts with the notation of the Master Theorem.) Here we continue to assume the obvious unstated assumptions of the Master Theorem.

Observe that T is asymptotically positive if the base case is non-negative and f is non-negative and asymptotically positive.

35. Applications to Nonhomogeneous Difference Equations

We now obtain an asymptotic formula for solutions of a large class of difference equations. A change of variables yields admissible recurrences amenable to our main results.

Theorem 35.1. Let n_0 and k be integers with $k > 0$. Define $D = \boldsymbol{Z} \cap [n_0 - k, \infty)$ and $I = \boldsymbol{Z} \cap [n_0, \infty)$. Suppose $f: D \backslash I \to \boldsymbol{R}$ and $g: I \to \boldsymbol{R}$ with f positive. Let $a_1, \dots, a_k$ be non-negative real numbers that are not all zero.

Let $I^* = \{e^n : n \in I\}$, and define $g^*: I^* \to \boldsymbol{R}$ by $g^*(s) = g(\log s)$ for all $s \in I^*$. Suppose g^* has polynomial growth. Let $C^*: [e^{n_0}, \infty) \to \boldsymbol{R}$ be a continuous, polynomial-growth extension of g^* (such a C^* exists by Lemma 5.1). Define $C: [n_0, \infty) \to \boldsymbol{R}$ by $C(t) = C^*(e^t)$ for all $t \in [n_0, \infty)$.

There exists exactly one real-valued function $T: D \to \boldsymbol{R}$ that satisfies $T|_{D \backslash I} = f$ and

$$T(n) = \sum_{j=1}^{k} a_j T(n-j) + g(n)$$

for all $n \in I$. There exist positive real numbers γ and δ such that

$$\gamma B(n) \leq T(n) \leq \delta B(n)$$

for all $n \in I$ where

$$B(n) = \lambda^n \left(1 + \int_{n_0}^{n} \frac{C(u)}{\lambda^u} du\right)$$

and λ is the unique positive root of the polynomial

$$x^k - \sum_{j=1}^{k} a_j x^{k-j}.$$

If $g(n) = O(\lambda^n / n^{1+\varepsilon})$ for some $\varepsilon > 0$, then $T(n) = \Theta(\lambda^n)$.

Proof. The recurrence is finitely recursive and therefore has a unique solution T by Lemma 8.2. Let

$$E = \{j \in \boldsymbol{Z} \cap [1,k] : a_j \neq 0\}.$$

Lemma 11.1 implies there exists exactly one real number p that satisfies

$$\sum_{j \in E} a_j e^{-jp} = 1,$$

i.e.,

$$\sum_{j=1}^{k} a_j e^{-jp} = 1.$$

The exponential function on $\boldsymbol{R}$ is a bijection onto the set of positive real numbers, so there exists exactly one real positive number l that satisfies

$$\sum_{j=1}^{k} a_j l^{-j} = 1.$$

Furthermore, $l = e^p$ and l is the unique positive root of the polynomial

$$x^k - \sum_{j=1}^{k} a_j x^{k-j},$$

i.e., $\lambda = e^p$. (Existence of exactly one positive root for the polynomial also follows from Descartes's rule of signs, which also implies the root is simple. See [Us].)

Let $D^* = \{e^n : n \in D\}$, so $I^* = D^* \cap [e^{n_0}, \infty)$ and $D^* \backslash I^* = \{e^n : n \in D \backslash I\}$. Define $f^* : D^* \backslash I^* \to \boldsymbol{R}$ by $f^*(v) = f(\log v)$ for all $v \in D^* \backslash I^*$. The domain, $D \backslash I$, of f has finite cardinality k. Observe that $Range(f^*) = Range(f)$ is a positive set, which is finite. Therefore, f^* has a positive lower bound and a finite upper bound.

The function g^* is non-negative by Lemma 2.7. Since $g(n) = g^*(e^n)$ for all $n \in I$, we conclude that g is also a non-negative function (and is therefore eligible for asymptotic relationships according to our convention for asymptotic notation). Continuity of C^* implies C^* is locally Riemann integrable, so C^* is a tame extension of g^*.

Let R^* be the divide-and-conquer recurrence

$$T^*(x) = \begin{cases} f^*(x), & \text{for } x \in D^* \backslash I^* \\ \displaystyle\sum_{j \in E} a_j T^*(e^{-j} x) + g^*(x), & \text{for } x \in I^* \end{cases}$$

with domain D^*, recursion set I^*, base case, f^*, incremental cost g^*, and dependencies $x \mapsto e^{-j}x$ where j varies over the elements of E. The noise terms are identically zero, so R^* has low noise and is therefore admissible. The Akra-Bazzi exponent of R^* is p.

The recurrence R^* satisfies the bounded depth condition, so Corollary 20.12 implies R^* has a unique solution T^*, which satisfies the strong Akra-Bazzi condition relative to R^* and C^*. Of course,

$$T^*(x) = T(\log x)$$

for all $x \in D^*$, and

$$T(n) = T^*(e^n)$$

for all $n \in D$. There exists $\gamma, \delta > 0$ such that

$$\gamma A(x) \leq T^*(x) \leq \delta A(x)$$

for all $x \in I^*$, i.e.,

$$\gamma A(e^n) \leq T(n) \leq \delta A(e^n)$$

for all $n \in I$ where $A\colon I^* \to \boldsymbol{R}$ is defined by

$$A(x) = x^p \left(1 + \int_{e^{n_0}}^{x} \frac{C^*(z)}{z^{p+1}} dz\right)$$

for all $x \in I^*$, i.e.,

$$A(e^n) = e^{np} \left(1 + \int_{e^{n_0}}^{e^n} \frac{C^*(z)}{z^{p+1}} dz\right)$$

for all $n \in I$.

Continuity of C^* implies continuity of C. Then the function on $[n_0, \infty)$ that maps u to $C(u)/e^{pu}$ is also continuous, so integration by substitution of $u = \log z$ is justified to obtain

$$\int_{e^{n_0}}^{e^n} \frac{C^*(z)}{z^{p+1}} dz = \int_{n_0}^{n} \frac{C(u)}{e^{pu}} du$$

for all $n \in I$, which combines with $\lambda = e^p$ to imply $A(e^n) = B(n)$ and

$$\gamma B(n) \leq T(n) \leq \delta B(n)$$

for each such n. (In particular, $T(n) = \Theta\big(B(n)\big)$.)

Now suppose $g(n) = O(\lambda^n/n^{1+\varepsilon})$ for some $\varepsilon > 0$. By Lemma 2.7, either C^* is a positive function or C^* is identically zero. If C^* is identically zero, then C is identically zero and

$$B(n) = \lambda^n$$

for all $n \in I$, so

$$T(n) = \Theta(\lambda^n).$$

Therefore, we may assume C^* is a positive function, so $\Psi_e(C^*)$ is defined. Lemmas 2.10(2) and 2.16 imply $1 \le \Psi_e(C^*) < \infty$. Positivity of C^* implies positivity of C.

There exists a positive element m of I and a positive real number L such that $g(n) \le L\,\lambda^n/n^{1+\varepsilon}$ for each integer $n \ge m$. Define

$$H = 2^{1+\varepsilon} \cdot \max\{1, 1/\lambda\} \cdot \Psi_e(C^*)L$$

and let $w \in [m, \infty)$, so $\lfloor w \rfloor \ge m \ge 1$. Since $w \in [\lfloor w \rfloor. \lfloor w \rfloor + 1)$, we have

$$e^w \in \left[e^{\lfloor w \rfloor}, e^{\lfloor w \rfloor + 1}\right).$$

Lemma 2.10(4) implies

$$C^*(e^w) \le \Psi_e(C^*)C^*\left(e^{\lfloor w \rfloor}\right),$$

i.e.,

$$C(w) \le \Psi_e(C^*)C(\lfloor w \rfloor).$$

Observe the $\lfloor w \rfloor - w \in [-1,0]$ and

$$0 < w/\lfloor w \rfloor < (\lfloor w \rfloor + 1)/\lfloor w \rfloor \le 2,$$

so

$$\frac{\lambda^{\lfloor w \rfloor}/\lfloor w \rfloor^{1+\varepsilon}}{\lambda^w/w^{1+\varepsilon}} \;=\; \lambda^{\lfloor w \rfloor - w}\left(\frac{w}{\lfloor w \rfloor}\right)^{1+\varepsilon} \;<\; \frac{H}{\Psi_e(C^*)L}.$$

We conclude from $C(\lfloor w \rfloor) = g(\lfloor w \rfloor)$ and $\lfloor w \rfloor \ge m$ that

$$C(w) \;\le\; \Psi_e(C^*)g(\lfloor w \rfloor) \;\le\; \Psi_e(C^*)L\frac{\lambda^{\lfloor w \rfloor}}{\lfloor w \rfloor^{1+\varepsilon}} \;<\; H\frac{\lambda^w}{w^{1+\varepsilon}}$$

(so $C(x) = O(\lambda^x/x^{1+\varepsilon})$), which combines with non-negativity of C to imply

$$0 \le \int_m^n \frac{C(u)}{\lambda^u}du \le H\int_m^n \frac{1}{u^{1+\varepsilon}}du = \frac{H}{\varepsilon}\left(\frac{1}{m^\varepsilon} - \frac{1}{n^\varepsilon}\right) < \frac{H}{\varepsilon m^\varepsilon}$$

and

$$0 < 1 + \int_{n_0}^m \frac{C(u)}{\lambda^u}du \le 1 + \int_{n_0}^n \frac{C(u)}{\lambda^u}du \le 1 + \int_{n_0}^m \frac{C(u)}{\lambda^u}du + \frac{H}{\varepsilon m^\varepsilon}$$

for each integer $n \ge m$. Then

$$1 + \int_{n_0}^n \frac{C(u)}{\lambda^u}du = \Theta(1),$$

which implies

$$T(n) = \Theta(\lambda^n).$$

□

Corollary 35.2. Let n_0 and k be positive integers. Define $D = \boldsymbol{Z} \cap [n_0 - k, \infty)$ and $I = \boldsymbol{Z} \cap [n_0, \infty)$. Suppose $f: D\backslash I \to \boldsymbol{R}$ is positive and $g: I \to \boldsymbol{R}$ has polynomial growth. Let $a_1, \dots, a_k$ be non-negative real numbers that are not all zero.

There exists exactly one real-valued function $T: D \to \boldsymbol{R}$ that satisfies $T|_{D\backslash I} = f$ and

$$T(n) = \sum_{j=1}^{k} a_j T(n-j) + g(n)$$

for all $n \in I$. There exist tame extensions of g. Furthermore,

$$T(n) = \Theta\left(\lambda^n \left(1 + \int_{n_0}^{n} \frac{G(u)}{\lambda^u}\, du\right)\right)$$

for each tame extension G of g where λ is the unique positive root of the polynomial

$$x^k - \sum_{j=1}^{k} a_j x^{k-j}.$$

If any of the following three conditions is satisfied, then $T(n) = \Theta(\lambda^n)$:

(1) g is identically zero.

(2) $\lambda > 1$.

(3) $\lambda = 1$ and $g(n) = O(1/n^{1+\varepsilon})$ for some $\varepsilon > 0$.

Proof. The recurrence is finitely recursive and therefore has a unique solution T by Lemma 8.2. Lemma 2.7 implies g is non-negative (and is therefore eligible for asymptotic relationships according to our convention for asymptotic notation).

By Descartes's rule of signs (see [Us]), there is indeed exactly one positive root, λ, of the polynomial

$$x^k - \sum_{j=1}^{k} a_j x^{k-j}.$$

By Corollary 5.2, there exists a continuous, polynomial-growth function $C: [n_0, \infty) \to \boldsymbol{R}$ with $C|_I = g$: Continuity of C implies C is locally Riemann integrable and is therefore a tame function. In particular, g has at least one tame extension to $[n_0, \infty)$. Let G be any such extension.

Let $I^* = \{e^n : n \in I\}$, and define $g^*: I^* \to \boldsymbol{R}$ by $g^*(s) = g(\log s)$ for all $s \in I^*$. Define $C^*: [e^{n_0}, \infty) \to \boldsymbol{R}$ by $C^*(r) = C(\log r)$ for all $r \in [e^{n_0}, \infty)$. Observe that $C^*|_{I^*} = g^*$ and $C(t) = C^*(e^t)$ for all $t \in I$. The functions g^* and C^* have polynomial growth by Lemmas 4.1(3) and 4.6. The function C^* is continuous.

Lemmas 2.7 and 2.34 imply $g(n) = O(\lambda^n/n^{1+\varepsilon})$ for some $\varepsilon > 0$ if (and only if) at least one of the conditions (1), (2), and (3) is satisfied. (There is no condition listed for $\lambda < 1$ because when $\lambda < 1$, the relationship $g(n) = O(\lambda^n/n^{1+\varepsilon})$ holds if and only if g is identically zero, i.e., condition (1) is satisfied—see Lemma 2.34.) In particular, if one of the conditions (1), (2), and (3) is satisfied, then $T(n) = \Theta(\lambda^n)$ by Theorem 35.1.

By Theorem 35.1, there exist positive real numbers γ and δ such that

$$\gamma B(n) \leq T(n) \leq \delta B(n)$$

for all $n \in I$ where $B: I \to \boldsymbol{R}$ is defined by

$$B(n) = \lambda^n \left(1 + \int_{n_0}^{n} \frac{C(u)}{\lambda^u}\, du\right).$$

In particular, $T(n) = \Theta\big(B(n)\big)$. Define $A: I \to \boldsymbol{R}$ by

$$A(n) = \lambda^n \left(1 + \int_{n_0}^{n} \frac{G(u)}{\lambda^u}\, du\right)$$

for all $n \in I$. Lemma 2.7 and

$$G(n_0) = g(n_0) = C(n_0)$$

imply either all of g, G and C are positive function or all of them are identically zero. In particular, they are all non-negative. If G and C are identically zero, then $A = B$ and $T(n) = \Theta\big(A(n)\big)$. Therefore, we may assume G and C are positive functions. The domains of G and C are sets of positive real numbers, so $\Psi_2(G)$ and $\Psi_2(C)$ are defined. Lemma 2.10(2) implies $\Psi_2(G) \geq 1$ and $\Psi_2(C) \geq 1$. Lemma 2.16 implies $\Psi_2(G) < \infty$ and $\Psi_2(C) < \infty$.

Suppose $u \in [n_0, \infty)$, and let $t = \lfloor u \rfloor$, so

$$u \in [t, t+1) \subset [t, 2t] \subset [n_0, \infty).$$

Lemma 2.10(4) implies

$$\frac{G(u)}{\Psi_2(G)} \leq G(t) \leq \Psi_2(G)G(u)$$

and

$$\frac{C(u)}{\Psi_2(C)} \leq C(t) \leq \Psi_2(C)C(u).$$

We conclude from

$$G(t) = g(t) = C(t)$$

that

$$\frac{G(u)}{\Psi_2(C)\Psi_2(G)} \le \frac{C(t)}{\Psi_2(C)} \le C(u) \le \Psi_2(C)C(t) \le \Psi_2(C)\Psi_2(G)G(u).$$

Define

$$\alpha = \frac{\gamma}{\Psi_2(C)\Psi_2(G)}$$

and

$$\beta = \delta\Psi_2(C)\Psi_2(G),$$

so that $0 < \alpha \le \gamma$ and $\beta \ge \delta > 0$. Then

$$\alpha G(u) \le \gamma C(u) \text{ and } \delta C(u) \le \beta G(u).$$

Therefore,

$$\alpha A(n) \le \gamma B(n) \le T(n) \le \delta B(n) \le \beta A(n)$$

for all $n \in I$, so

$$T(n) = \Theta\big(A(n)\big).$$

□

Recurrences satisfying either the hypothesis of Theorem 35.1 with $n_0 > 0$ or the hypothesis of Corollary 35.2 (which implies the hypothesis of Theorem 35.1 with $n_0 > 0$) also satisfy our overly loose definition of a divide-and-conquer recurrence if we ignore the terms $a_j(n-j)$ with $a_j = 0$. Such recurrences are inadmissible because of their high noise.

Variations on the Fibonacci Numbers. Let $g\colon \boldsymbol{Z} \cap [3,\infty) \to \boldsymbol{R}$ be a polynomial-growth function. Define $T\colon \boldsymbol{Z}^+ \to \boldsymbol{R}$ by the recurrence

$$T(n) = \begin{cases} 1, & \text{for } n \in \{1,2\} \\ T(n-1) + T(n-2) + g(n), & \text{for } n \ge 3. \end{cases}$$

By Corollary 35.2,

$$T(n) = \Theta(\varphi^n)$$

where

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6$$

is the positive root of the polynomial $x^2 - x - 1$, i.e., φ is the golden ratio. The other root is

$$\psi = \frac{1-\sqrt{5}}{2} \approx -0.6.$$

If g is identically zero, then $T(n)$ is the nth Fibonacci number F_n. The well-known formula

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

(explained in Section 7) is consistent with $F_n = \Theta(\varphi^n)$ because $\varphi > |\psi|$.

An example from *generatingfunctionology*. The simple recurrence

$$T(n) = \begin{cases} 1, & \text{for } n = 0 \\ 2T(n-1) + n - 1, & \text{for } n > 0 \end{cases}$$

with domain $\boldsymbol{N}$ has the solution

$$T(n) = 2^{n+1} - n - 1,$$

which can be derived via a generating function ([Wilf, pp. 5-7]. The solution can also be guessed by examining the first few terms of the sequence

$$T(0), T(1), T(2) \ldots = 1, 2, 5, 12, 27, 58, 121, \ldots$$

and proved by induction. Observe that

$$2^n \leq T(n) < 2 \cdot 2^n$$

for all $n \in \boldsymbol{N}$, so $T(n) = \Theta(2^n)$.

The function T is also the solution of the (equivalent) recurrence

$$T(n) = \begin{cases} 1, & \text{for } n = 0 \\ 2, & \text{for } n = 1 \\ 2T(n-1) + n - 1, & \text{for } n \geq 2. \end{cases}$$

The incremental cost is the function $n \mapsto n - 1$ on $\boldsymbol{Z} \cap [2, \infty)$, which has polynomial growth by Lemma 4.7. The polynomial $x - 2$ has the unique root 2, so Corollary 35.2 agrees that $T(n) = \Theta(2^n)$.

Example with $\boldsymbol{\lambda = 1}$ and $\boldsymbol{T(n) = \Theta(1)}$. Define $g: \boldsymbol{Z} \cap [2, \infty) \to \boldsymbol{R}$ by

$$g(n) = \frac{1}{n\sqrt{n}}.$$

Lemma 4.1(2) implies g has polynomial growth. Define $T: \boldsymbol{N} \to \boldsymbol{R}$ by

$$T(n) = \begin{cases} 1, & \text{for } n \in \{0,1\} \\ \frac{1}{2}T(n-1) + \frac{1}{2}T(n-2) + g(n), & \text{for } n \geq 2. \end{cases}$$

The polynomial

$$x^2 - \frac{1}{2}x - \frac{1}{2}$$

has positive root $\lambda = 1$. Observe that

$$g(n) = 1/n^{1+\varepsilon} = O(1/n^{1+\varepsilon})$$

for $\varepsilon = 1/2$. Corollary 35.2 implies $T(n) = \Theta(\lambda^n)$, i.e., $T(n) = \Theta(1)$.

Example with $\boldsymbol{\lambda = 1}$ and $\boldsymbol{T(n) \neq \Theta(1)}$. The functions $g: \boldsymbol{Z} \cap [2, \infty) \to \boldsymbol{R}$ and $G: [2, \infty) \to \boldsymbol{R}$ defined by

$$g(n) = \frac{1}{n}$$

for all $n \in \boldsymbol{Z} \cap [2, \infty)$ and

$$G(x) = \frac{1}{x}$$

for all $x \in [2, \infty)$ have polynomial growth by Lemma 4.1(2). Furthermore, g is the restriction of G to $\boldsymbol{Z} \cap [2, \infty)$, the function G is locally Riemann integrable, and $domain(G)$ is a positive interval, so G is a tame extension of g. Observe that

$$g(n) \neq O(1/n^{1+\varepsilon})$$

for all $\varepsilon > 0$. Define $T: \boldsymbol{N} \to \boldsymbol{R}$ by

$$T(n) = \begin{cases} 1, & \text{for } n \in \{0,1\} \\ \frac{1}{3}T(n-1) + \frac{2}{3}T(n-2) + g(n), & \text{for } n \geq 2. \end{cases}$$

The polynomial

$$x^2 - \frac{1}{3}x - \frac{2}{3}$$

has the positive root $\lambda = 1$. Corollary 35.2 implies

$$T(n) = \Theta\left(1^n\left(1 + \int_{n_0}^{n} \frac{1/u}{1^u}\,du\right)\right) = \Theta\left(1 + \int_{n_0}^{n} \frac{1}{u}\,du\right),$$

so $T(n) = \Theta(\log n)$.

Example with $\boldsymbol{\lambda < 1}$ and $\boldsymbol{T(n) \neq \Theta(\lambda^n)}$. Let the functions $g: \boldsymbol{Z} \cap [2, \infty) \to \boldsymbol{R}$ and $G: [2, \infty) \to \boldsymbol{R}$ be the identity functions defined by

$$g(n) = n$$

for all $n \in \boldsymbol{Z} \cap [2, \infty)$ and

$$G(x) = x$$

for all $x \in [2, \infty)$. The functions g and G have polynomial growth by Lemma 4.1(2). Of course, G is a tame extension of g. Define $T: \boldsymbol{N} \to \boldsymbol{R}$ by

$$T(n) = \begin{cases} 1, & \text{for } n \in \{0,1\} \\ \frac{1}{4}T(n-1) + \frac{1}{8}T(n-2) + n, & \text{for } n \geq 2. \end{cases}$$

The polynomial

$$x^2 - \frac{1}{4}x - \frac{1}{8}$$

has the positive root $\lambda = 1/2$. Corollary 35.2 implies

$$T(n) = \Theta\left(\frac{1}{2^n}\left(1 + \int_{n_0}^{n} 2^u u du\right)\right).$$

We conclude from

$$\int_{n_0}^{n} 2^u u du = \frac{2^n(n\,log(2) - 1)}{\log^2 2} - \frac{2^{n_0}(n_0\,log(2) - 1)}{\log^2 2}$$

that

$$T(n) = \Theta(n).$$

Alternatively, we can simply notice and prove by induction that

$$n \leq T(n) < \frac{8}{5}n$$

when $n > 0$.

Example with exponential incremental cost. Let $I = \boldsymbol{Z} \cap [2, \infty)$. Define $g: I \to \boldsymbol{R}$ and $C: [2, \infty) \to \boldsymbol{R}$ by $g(n) = e^n$ and $C(x) = e^x$. Define $T: \boldsymbol{N} \to \boldsymbol{R}$ **by**

$$T(n) = \begin{cases} 1, & \text{for } n \in \{0,1\} \\ 4T(n-1) + 5T(n-2) + e^n, & \text{for } n \geq 2. \end{cases}$$

In the language of Theorem 35.1, $I^* = \{e^n : n \in \boldsymbol{Z} \cap [2, \infty)\}$; the function g^* is the identity function on I^*. Let C^* be the identity function on $[e^2, \infty)$, so C^* is a continuous extension of g^*. The functions g^* and C^* have polynomial growth by Lemma 4.1(2). Observe that $C(t) = C^*(e^t)$ for all $t \in [2, \infty)$.

The polynomial

$$x^2 - 4x - 5$$

has the positive root 5. Theorem 35.1 implies

$$T(n) = \Theta\left(5^n\left(1 + \int_{n_0}^{n} \frac{e^u}{5^u}\, du\right)\right).$$

Observe that $e < 5$, so $(e/5)^n$ approaches 0 as n approaches ∞. Therefore,

$$\int_{n_0}^{n} \frac{e^u}{5^u} u du = \frac{\left(\frac{e}{5}\right)^n - \left(\frac{e}{5}\right)^{n_0}}{\log\left(\frac{e}{5}\right)} = \frac{\left(\frac{e}{5}\right)^{n_0} - \left(\frac{e}{5}\right)^n}{\left|\log\left(\frac{e}{5}\right)\right|} = \Theta(1),$$

which implies

$$T(n) = \Theta(5^n).$$

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